Equilibrium Price Dynamics of Emission Permits

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Abstract

This paper presents a stochastic equilibrium model for environmental markets that allows us to study the characteristic properties of emission permit prices induced by the design of today’s cap-and-trade systems. We characterize emission permits as highly nonlinear contingent claims on economy-wide emissions and reveal their hybrid nature between investment and consumption assets. Our model makes predictions about the dynamics and volatility structure of emission permit prices, the futures price curve, and the implications for option pricing in this market. Empirical evidence from existing emissions markets shows that the model explains the stylized facts of emission permit prices and related derivatives.

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1 Introduction

Global warming is generally accepted as one of the greatest challenges that society faces in the 21st century, bringing the goal to limit and reduce anthropogenic greenhouse gas emissions to the top of political agendas worldwide. One of the most promising approaches is the use of market-based instruments, which are undoubtedly a key contribution of economics (Stavins 2011). Based on the seminal work of Coase (1960), the idea is to oblige polluters to acquire a right for each ton of greenhouse gas they emit, and to limit the amount of available rights according to a predefined cap. Starting from an initial allocation, these rights, called emission permits, can be traded among the polluters and other investors. Under perfect market conditions this market mechanism leads to an efficient allocation of permits such that the emission limit is achieved at lowest cost for the economy. In the recent years, the attention given to such market-based instruments both by policy-makers and regulators has materially increased. As a consequence, mandatory emission trading schemes are introduced on international, national, or domestic levels all over the world — in the European Union, for several states of the US, in New Zealand and Australia, for the metropolitan area of Tokyo — and with them, a sizeable financial market has evolved. The global transaction volume related to emissions markets amounts to USD 176 billion in 2011, with a large share falling on derivatives written on emission permits which are traded on several exchanges around the world.

While the environmental economics literature has improved our understanding of how to design sensible, cost-effective policy solutions, it is silent when it comes to pricing and hedging emissions-related contingent claims. General practice is to apply models for classical financial assets or for commodity prices, although there are reasonable doubts that these models adequately reflect emission permit price characteristics. Indeed, there is even disagreement whether to categorize them as commodities or as financial assets. Against this background this paper presents a stochastic equilibrium model for emissions markets that allows us to study the characteristic properties of emission permit prices originating from today’s emission trading systems. We characterize emission permits as highly nonlinear contingent

1See the State and Trends of the Carbon Market report of the World Bank authored by Kossoy and Guigon (2012). According to the authors, this figure is based on hand-collected transaction data from various sources and includes virtually all relevant emission trading systems worldwide.
claims on economy-wide emissions, with important implications for their price dynamics and volatility structure. On top of this, the underlying is non-tradable and cannot be considered as exogenously given due to the emission reduction measures implemented by regulated polluters. Furthermore, the model reveals the hybrid nature of emission permits between investment and consumption assets, resulting in a futures price curve that is partly in normal contango and partly backwardated. Market data from the most important emission trading systems consistently supports our theoretical findings.

Our results are of importance for at least three categories of economic agents. First, polluting companies are exposed to the price risk of emission permits as a new production factor for their businesses and engage in derivatives markets to enhance their risk management. Second, emission permits and related derivatives also provide a promising option for diversification to financial investors and appear as a natural hedge against the temperature risk priced in equity returns (Bansal and Ochoa 2012). Third, it is of great interest for policy-makers and regulators to understand the relation of the system’s design and the induced stochastic behavior of emission permit prices, which is a key determinant of polluting companies’ investments into environment-friendly technologies.²

Our model considers an economy of companies that is regulated by an emission trading system. Companies have stochastic greenhouse gas emissions which they can reduce by implementing costly abatement measures, and they receive an initial amount of emission permits that are tradable in the market. We particularly account for the design features of today’s state-of-the-art emission trading systems, which are organized as a number of consecutive compliance periods. At the end of each compliance period, the regulated companies are obliged to cover their emissions with emission permits, and a penalty is incurred in case of non-compliance. Additionally, companies have to deliver the lacking permits in the following compliance period. While it is allowed to transfer unused permits from one period to the following one (banking), the reverse direction (borrowing) is prohibited. Given these rules, each company optimizes its abatement and trading strategy to find an optimal trade-off between implementing abatement measures, trading permits in the market, and taking the risk of penalty payments.

The marginal value of an emission permit in equilibrium derives from the specific design of

²The dependency of investment decisions on the underlying asset price dynamics is well studied in the literature on classical commodities (e.g., Brennan and Schwartz 1985) and real options (e.g., Pindyck 1988).
the emission trading system. Particularly, we show that an emission permit can be characterized as a strip of European binary options written on economy-wide emissions. In contrast to classical financial options, however, the dynamics of this non-tradable underlying is not exogenously given, since companies can and obviously do influence it through their endogenously derived abatement measures. Abatement reduces the companies’ greenhouse gas emissions, such that an emission permit is worth less than the corresponding strip of binary options in a scenario where no abatement is possible. We exploit this option analogy to disclose the characteristic properties of the permit price dynamics induced by the system’s design: First, emission permits exhibit a state- and time-dependent volatility structure. Similar to the ‘leverage effect’ for stocks as options on the firm value (Geske 1979), we obtain a negative relation between permit prices as (strips of binary) options on economy-wide emissions and their volatilities.

Second, our model predicts distinctive features of the futures price curve in emission permit markets. Emission permit futures with maturity in the next compliance period cannot be used for compliance in the ongoing period, such that holding the spot emission permit provides an additional benefit to the owner as we know it from classical commodities. To the contrary, there is no such advantage compared to futures contracts maturing in the same period, since companies need to cover their emissions with permits only at the very end of the compliance period. In other words, the usage option embedded in the spot asset (Routledge et al. 2000) can only be exercised at the end of a compliance period and emission permits are pure investment assets before that date. This hybrid nature between investment and consumption assets leads to a futures price curve that is in contango within compliance periods, but generally backwardated across them. As a consequence, instantaneous convenience yield models often used for classical commodities cannot consistently describe the futures price curve of emission permits.

Third, a calibrated version of our model in line with the Phase II (the second compliance period from 2008 to 2012) of the EU Emissions Trading System (EU ETS) reveals more detailed features of permit prices and local volatilities in the world’s largest emission trading system. An extensive simulation study shows that the volatility smile of options on emission permits is downward-sloping for most realistic emissions scenarios, which is in direct accordance with the negative relation of permit prices and volatilities explained above and thus also a result of the permit’s payoff structure induced by the system’s design.
We verify the model predictions using data from two important emissions markets, in particular from Phase II of the EU ETS as well as from the US Acid Rain Program for sulfur dioxide emissions. This data confirms all our main testable predictions: the negative relation of permit prices and volatilities, the shape of the futures price curve within and across compliance periods, and the downward-sloping volatility smile in emission permit markets.

Our paper contributes to the connection of two originally separate strands of literature. In the financial literature on commodity markets and the assessment of commodity-related securities and projects (Schwartz 1997; Routledge et al. 2000; Schwartz and Smith 2000; Casassus and Collin-Dufresne 2005; Trolle and Schwartz 2009), it is well recognized that economic decisions such as risk management and investment decisions depend critically on the assumed stochastic behavior of commodity prices. This research naturally focusses on classical commodities like oil, gas, or agricultural goods, and it is at least questionable that the existing approaches directly transfer to the novel commodity of emission permits.

On the other hand, the environmental economics literature spends much attention to the analysis of equilibrium outcomes within emission trading systems dependent on important design features as well as the comparison to conventional command-and-control mechanisms (Montgomery 1972; Cronshaw and Kruse 1996; Rubin 1996; Schennach 2000). Theoretical works in this area, however, build on a deterministic model framework, which precludes to investigate economic questions that arise in consequence of price uncertainty.

A newly evolving literature links these two strands by developing equilibrium models for permit markets under uncertainty. A few studies investigate the optimal strategies of agents and permit price properties within a stochastic framework using a stylized representation of an emission trading system that abstracts from the specific design features of today’s systems. In particular, these approaches consider emissions trading in a setting of one finite compliance period (Seifert et al. 2008; Carmona et al. 2009, 2010; Chesney and Taschini 2012) or under symmetric rules for banking and borrowing (Kijima et al. 2010; Cetin and Verschuere 2009). Since the value of emission permits only arises due to the regulatory rules of the system, we argue that it is crucial to account for the specific design of today’s emission trading systems to make clear-cut predictions about the stochastic behavior of permit prices and to derive implications for related derivatives. Indeed, this paper shows that the design of state-of-the-art emission trading systems induces a number of characteristic features that are represented by emission permit spot, futures, and option prices.
The paper is organized as follows. Section 2 outlines the current state of emissions trading around the world. In Section 3 we introduce our stochastic equilibrium model. Section 4 derives equilibrium outcomes and characterizes the price dynamics of emission permits, the futures price curves, and applies the model to the valuation of options in this market. Section 5 provides empirical evidence for the predictions of our model. Section 6 concludes.

2 Greenhouse Gas Emissions Trading

Greenhouse gas emissions trading has been established on a global level by the Kyoto Protocol, which was adopted in December 1997. The Kyoto Protocol assigns limited amounts of greenhouse gas emissions permitted for the years from 2008 to 2012 to the participating countries. Countries are allowed to transfer parts of their assigned amount to other countries, such that they do not necessarily need to comply with their individual emission limits as long as the aggregate emissions do not exceed the overall cap. On a smaller scale, there are examples of successfully implemented emission trading systems also before the Kyoto Protocol, most importantly the US Acid Rain Program for sulfur dioxide emissions which was in operation from 1995 to 2012. Nonetheless, the introduction of the Kyoto Protocol has clearly stimulated the motivation of governments worldwide to introduce emission trading systems on an international, national, or domestic level. We review the current state of greenhouse gas emissions trading worldwide.

Existing and Proposed Systems. Emission trading systems are by now in operation on almost all continents of the world. In Europe, the EU ETS was launched in 2005 and includes emissions of the 27 EU member states plus Iceland, Liechtenstein, and Norway. In the United States, mandatory emissions trading has been introduced by the Regional Greenhouse Gas Initiative (RGGI) in 2009 for nine states along the East Coast, and in California, where the California Emissions Trading Scheme (CA ETS) is in operation since the beginning of 2013. Further notable systems are the New Zealand Emissions Trading Scheme (NZ ETS), which exists since 2010, and the Australian Carbon Pricing Mechanism (CPM) launched in 2012. The EU ETS is by far the most significant one of these systems, followed by the CA ETS. More or less concrete plans to establish emissions trading exist in many other parts of

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3 The RGGI member states are Connecticut, Delaware, Maine, Maryland, Massachusetts, New Hampshire, New York, Rhode Island, and Vermont.
the world. Most notably, the Republic of Korea passed a bill that establishes a nation-wide emissions trading scheme by 2015. Countries like China, Japan, and Brazil underpin their ambitions to establish emissions trading by setting up domestic schemes on a mandatory or voluntary basis.

**Design and Market Mechanism.** All notable existing and proposed emission trading systems are designed in a very similar fashion according to the *cap-and-trade* paradigm. In particular, they are organized as a number of consecutive compliance periods for which the regulator sets a cap on the emission volume and allocates 4 a corresponding amount of emission permits to the regulated entities. Companies are obliged to cover their emissions by the end of each compliance period with a sufficient number of permits, which are then removed from the system and cannot be used again. To enforce compliance, a monetary penalty is imposed for each ton of exceeding emissions, plus the liability to deliver lacking permits in the next compliance period. Leftover permits allocated for the ongoing period can be carried over to following compliance periods (*banking*), while it is prohibited to use permits from future allocations earlier (*borrowing*). Given these rules, companies face the challenge of minimizing the costs incurred to them by the emissions trading system, and seek for an optimal strategy of implementing emission abatement measures, buying or selling permits in the market, and taking the risk of penalty payments. As a result, polluters with relatively low marginal abatement costs perform abatement actions in excess of their individual compliance goal and sell additional permits in the market, while polluters with relatively high marginal abatement costs buy these permits and achieve compliance with less individual abatement. According to Montgomery (1972), this mechanism leads to an efficient allocation of abatement measures under perfect market conditions, such that emission reductions are achieved at lowest cost for the economy.

**Emissions and Abatement.** It is common to describe scenarios of a company’s or the economy’s future emissions by means of business-as-usual emissions that arise under the assumption of no abatement activities, and the available abatement opportunities that yield emission reductions compared to the business-as-usual case. The worldwide business-as-usual greenhouse gas emissions amount to 50.1 gigatons of carbon dioxide (CO₂) equivalent 5

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4 Permits are either distributed for free or auctioned to the polluting companies.

5 Greenhouse gas emissions are usually expressed in CO₂ equivalent, i.e., the amount of all greenhouse gases measured by their accelerating effect on global warming is translated to the amount of CO₂ that would cause the same effect when no other greenhouse gases are emitted to the atmosphere.
per year in 2010 and are expected to increase to a level of 58 gigatons by 2020.\footnote{Data on current and expected global greenhouse gas emissions and the abatement potential is based on the \textit{Emissions Gap Report} published by the \textit{United Nations Environment Programme} (2012).} About half of these emissions are produced by the energy and industrial sector, while the other half is related to land-use and consumer-related sectors like transport and waste management. The available abatement potential is estimated to about 17 gigatons per year by 2020, which means a reduction by about 29 percent compared to the business-as-usual scenario. \textit{Nauclér and Enkvist (2009)} classify possible abatement into measures increasing the energy efficiency, shifts to a less emission intensive power generation, and the preservation of sinks for greenhouse gases, especially forest and soils. The availability of abatement opportunities at different costs leads to the construction of marginal abatement cost curves describing the cost of one additional ton of emissions dependent on the volume of emission reductions compared to the business-as-usual case. Detailed bottom-up studies (Klepper and Peterson 2006; Nauclér and Enkvist 2009) show that marginal abatement costs increase at least linearly, and that the transition from one marginal cost level to the other is virtually continuous due to the diversity of available abatement measures.

\textit{Trading}. Trading of emission permits and related derivatives (often summarized as \textit{carbon trading}) takes place in equal shares over-the-counter as well as on several exchanges worldwide. The world’s largest and most important trading venue is the European Climate Exchange (ECX) operated by IntercontinentalExchange (ICE), and several other prominent exchanges run carbon trading facilities by now, like the New York Mercantile Exchange (NYMEX) operated by CME Group, the Nasdaq OMX Commodities Europe, and the European Energy Exchange (EEX) as a subsidiary of the EUREX. Emission permits of the EU ETS, called European Union Allowances (EUAs), and related derivatives account for the largest share of the global transaction volume in emissions markets. The most liquidly traded instruments are EUA futures accounting for 88\% of the EUA transaction volume in 2011, while trading in spot EUAs accounts for only 2\% of the transactions.\footnote{See Kossoy and Guigon (2012).} EUA futures are available with maturities up to 2020, expiring quarterly on the last Monday of March, June, September, and December for the first years, and annually in December for the later maturities. In the same way, European options written on EUA futures with the same maturities are available and liquidly traded.
3 Theoretical Model

We consider an economy given by a set of companies $I$ whose emissions are regulated under a cap-and-trade system with $n$ consecutive compliance periods $[0,T_1], [T_1,T_2], \ldots, [T_{n-1},T_n]$. At time 0, each company $i \in I$ receives an endowment $(e^i_1, \ldots, e^i_n)$ of emission permits for the different periods of the system, i.e., $e^i_k$ is the initial amount of period-$k$ permits that are valid for compliance in period $[T_{k-1},T_k]$, with $T_0 = 0$. Companies are obliged to cover the emissions realized during period $k$ by a sufficient number of period-$k$ permits by the end of the period, $T_k$. For enforcement, a penalty of $p_k$ is imposed for each ton of exceeding emissions. In addition, lacking permits have to be delivered in the following compliance period, effectively reducing the number of period-$k+1$ permits. In a similar way, leftover permits not needed for compliance in period $k$ are banked into the next period, adding to the amount of period-$k+1$ permits. It is not allowed, however, to borrow permits that are allocated for a future compliance period and use them in the ongoing period. After $[T_{n-1},T_n]$, the last period of the system, left-over permits as well as obligations to later delivery are forfeited.

Let us consider these rules for a company $i$ whose emissions during the $n$ different compliance periods are specified by random variables $x^i_{0,T_1}, x^i_{T_1,T_2}, \ldots, x^i_{T_{n-1},T_n}$. For ease of illustration we first ignore the company’s abatement and trading activities. If $i$’s overall emissions during period 1 are larger than $e^i_1$, the exceeding $(x^i_{0,T_1} - e^i_1)$ tons of emissions are penalized, and $i$ has to deliver the lacking permits in period 2, decreasing the number of period-2 permits by the same amount. If, to the contrary, $x^i_{0,T_1} < e^i_1$, then $i$ banks the leftover period-1 permits into period 2, adding $(e^i_1 - x^i_{0,T_1})$ to the amount of period-2 permits. In summary, $i$ pays a penalty of $p_1(x^i_{0,T_1} - e^i_1)^+$ and the number of period-2 permits is altered to $e^i_2 + (e^i_1 - x^i_{0,T_1})$. Analyzing the same for period 2, we see that penalties are incurred if $x^i_{T_1,T_2} - (e^i_2 + e^i_1 - x^i_{0,T_1}) > 0$. This can be reinterpreted in the sense that the amount of exceeding emissions at the end of period 2 is the difference between $i$’s cumulative emissions from 0 up to $T_2$, $x^i_{0,T_2} = x^i_{0,T_1} + x^i_{T_1,T_2}$, and the cumulative amount of permits $q^i_2 = e^i_1 + e^i_2$ that is allocated for periods 1 and 2. In general terms, the amount of lacking or leftover permits in period $k$ results as the difference between $i$’s cumulative emissions up to $T_k$ and the cumulative allocation $q^i_k = \sum_{j=1}^{k} e^i_j$ for all compliance periods up to $k$, that is $x^i_{0,T_k} - q^i_k$. 
For convenience we simply write $x_{T_k}$ for $x_{0,T_k}$ throughout the rest of the paper. Overall, the present value of the expected penalty imposed on $i$ by the cap-and-trade system is

$$E_0 \left\{ \sum_{j=1}^{n} e^{-rT_j} p_j(x^i_{T_j} - q^i_j)^+ \right\},$$

(1)

where $r$ is the constant risk-free interest rate and $E_t \{ . \}$ denotes the expectation based on time-$t$ information.

Companies actively manage their risk of paying penalties by trading permits in the market and reducing their emissions through emission abatement measures. At each point in time, a company chooses the amount $\theta_{k,t}^i$ of period-$k$ permits it buys or sells in the market at equilibrium price $S_k(t)$, adjusting its time-$t$ holdings to $e_k^i + \int_0^t \theta_{s,t}^i ds$. Incorporating the trading strategy alters the expected penalty payment (1) by replacing $q^i_k$ with $Q^i_k = \sum_{j=1}^{k} \left( e_j^i + \int_0^{T_j} \theta_{j,s}^i ds \right)$. In the same way, the company decides about the operative abatement measures $\xi_t^i$ it implements at $t$ to reduce its instantaneous emissions. Without abatement, $i$’s emissions are driven by two components: A persistent business-as-usual emission rate $y^i$ following the Itô process

$$y^i_t = y^i_0 + \int_0^t \mu^i_y(s) ds + \int_0^t \sigma^i_y(s) dZ_s^i$$

(2)

with time-dependent drift $\mu^i_y(t)$ and volatility $\sigma^i_y(t) > 0$, and short-term emission shocks $n^i$ given by

$$n^i_t = \sigma^i_z \varepsilon^i_t$$

(3)

with $\sigma^i_z > 0$. $Z^i$ is a standard Wiener process and $\varepsilon^i$ is a standard Gaussian white noise process, and we assume that the increments of $W^i_t = \int_0^t \varepsilon^i_s ds$ and $Z^i_t$ are uncorrelated.\(^8\) While $y^i$ captures deterministic patterns of the emission rate like seasonality as well as fluctuations that are permanent in nature, $n^i$ represents temporary shocks like the outage of a carbon-friendly production unit that is instantaneously replaced by a more polluting one. Reduced by operative abatement $\xi_t^i$, $i$’s instantaneous emission rate is $y^i_t + n^i_t - \xi_t^i$, and its

\(^8\)We also assume that the increments of $W^i_t$ and $Z^i_t$ are uncorrelated across companies, i.e., for all $i, j \in I$. 

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cumulative emissions up to time $t$ result as

$$x_i^t = \int_0^t (y_s^i + n_s^i - \xi_s^i)ds. \quad (4)$$

Abating at an instantaneous rate of $\xi_i^t$ costs $C^i(\xi_i^t)$, where $C^i$ is a differentiable and convex abatement cost function, as motivated by detailed bottom-up studies (e.g., Klepper and Peterson 2006; Nauclé and Enkvist 2009) which point out that marginal abatement costs increase at least linearly.\(^9\)

Given these ingredients, each company maximizes its utility by finding an optimal trade-off between implementing abatement measures, trading permits in the market, and taking the risk of penalty payments. Under risk-neutrality,\(^10\) the company minimizes overall costs by solving the optimization problem

$$\min_{(\theta^i, \xi^i)} \mathbb{E}_0 \left\{ \int_0^{T_n} e^{-rt}C^i(\xi_i^t)dt + \sum_{j=1}^n \int_0^{T_j} e^{-rt}S_j(t)(\theta_j^t)dt + \sum_{j=1}^n e^{-rT_j}p_j(x_{T_j}^i - Q_{i,j})^+ \right\} \quad (5)$$

with optimal trading strategy $\theta^i = (\theta_i^1, \ldots, \theta_i^n)$, $\theta_k^i = (\theta_{k,t}^i)_{t \in [0,T_n]}$ and abatement strategy $\xi^i = (\xi_t^i)_{t \in [0,T_n]}$. The first term of (5) represents the costs for implementing abatement measures and the second term describes the costs of $i$'s trading strategy. The last term incorporates possible penalties at the end of each compliance period in accordance with (1).

\section{Emissions Market Equilibrium}

We solve the model for equilibrium permit prices that clear the market when all companies $i \in I$ choose optimal trading and abatement strategies according to (5). To begin with, we apply the stochastic maximum principle in conjunction with dynamic programming (see

\(^9\)Note that $C^i$ stands for the costs of operative abatement $\xi^i$, not for investments into carbon-friendly technologies. Carbon-related investments do not change the instantaneous emission rate, but lead to a flatter operative abatement cost function $C^i$ in the future, similar to general technological progress. Simple models for technological progress resulting in deterministically time-dependent abatement cost functions do not change the results of this paper. For optimal investment policies in the context of emission trading systems we refer to Taschini (2008) and the references therein.

\(^{10}\)The effects of risk aversion on the price dynamics of emission permits are analyzed by Seifert et al. (2008) within a framework of one single compliance period.
Appendix A) to characterize the optimal trading and abatement strategy of a company for given permit prices $S_1, \ldots, S_n$.

**Proposition 1 (Optimality Conditions).** For an optimal trading and abatement strategy $(\theta^i, \xi^i)$, a company’s instantaneous marginal abatement costs are equal to the permit price of the ongoing compliance period,

$$\frac{\partial C^i}{\partial \xi^i}(\xi^i_t) = S_k(t), \quad t \in [T_{k-1}, T_k]. \quad (6)$$

Further, the present value of a company’s expected overall penalty for an additional ton of emissions in period $k$ is equal to the price of period-$k$ emission permits, $k = 1, \ldots, n$, that is

$$\sum_{j=k}^{n} e^{-r(T_j-t)} P_t \{ x^{i,j}_{T_j} > Q_{j} \} p_j = S_k(t), \quad t \in [0, T_k]. \quad (7)$$

The first condition arises due to the fact that companies can achieve compliance equally by abating emissions or by buying additional permits in the market. Both actions reduce the number of uncovered emissions at the end of the ongoing compliance period. Consequently, the marginal cost of both actions is equal for an optimal strategy, since otherwise a company could improve by abating more emissions and buying less permits, or vice versa. This result is central to deterministic models for cap-and-trade systems developed in the environmental economics literature (e.g., Cronshaw and Kruse 1996; Rubin 1996).

The second condition characterizes the marginal value of an emission permit for an individual company, which is induced by the system’s penalty mechanism and the rules on banking, borrowing, and later delivery. If a company is short of permits for a particular compliance period $k$, then an additional permit saves a penalty payment and additionally also reduces the number of permits to be delivered later, which effectively increases the amount of permits in the next compliance period. In case of a permit surplus, an additional permit does not avoid any penalty payment, but it can be banked and adds to the available amount of permits in the following compliance period as well. The same logic applies for the following compliance

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$^{11}$ $P_t \{ \cdot \}$ denotes the probability conditional on time-$t$ information.

$^{12}$ This result is also shown within stochastic equilibrium models for one compliance period in the literature, see for example Seifert et al. (2008) and Carmona et al. (2009, 2010). In the context of multiple compliance periods, it is important to note that marginal abatement costs are equal to the permit price for the period in which the abatement is actually realized, which is the ongoing period in our case.
periods again. Overall, the marginal value of a period-\(k\) emission permit equals the sum of penalties for period \(k\) and all following compliance periods weighted by the probability that penalties arise. For an optimal strategy, this marginal value is equal to the permit price \(S_k\) for each company.\(^{13}\)

### 4.1 Permit Prices

Proposition 1 implies that in equilibrium, marginal abatement costs as well as the probability of penalties to accrue are equalized over all companies in the economy, in line with the emissions market mechanism first formalized by Montgomery (1972). In a situation where marginal abatement costs differ across companies, companies with lower marginal abatement costs extend their abatement activities and sell emission permits to companies with higher abatement costs, enabling them to cut back their abatement actions. If they agree on a permit price between their respective marginal abatement costs, both companies are able to profit from these actions, implying that such situation cannot persist in equilibrium. In the same way, the market is not in equilibrium if some companies have uncovered emissions at the end of a compliance period \(T_k\) while others have remaining permits. Taking a global point of view, this condition implies that one and thus every company’s emissions exceed their permit holdings exactly when economy-wide cumulative emissions \(x_{T_k} = \sum_{i \in I} x_{T_k}^i\) exceed economy-wide permit holdings \(Q_k = \sum_{i \in I} Q_k^i\). Since the market clearing condition holds in equilibrium, individual trading strategies cancel out in \(Q_k\) and we have \(Q_k = q_k = \sum_{i \in I} q_k^i\).

Altogether, the following result arises from condition (7).

**Proposition 2 (Equilibrium Permit Prices).** In equilibrium, the price of a period-\(k\) emission permit is given by

\[
S_k(t) = \sum_{j=k}^{n} e^{-r(T_j-t)} P_t \{ x_{T_j} > q_j \} p_j
\]

for \(t \in [0, T_k]\). That is, an emission permit is a strip of European binary call options written on cumulative economy-wide emissions.

This proposition characterizes emission permits created in the context of environmental trading schemes as a financial derivative. The price of an emission permit consists of one

\(^{13}\)For the special case of one single compliance period, this result is shown by Carmona et al. (2009, 2010).
value component for each compliance period of the system representing the probability that penalties accrue for that period because the economy is short of emission permits. At the end of the period, $T_k$, this value is equal to the penalty for exceeding emissions in that period if cumulative emissions $x_{T_k}$ exceed the cumulative allocation $q_k$, and zero otherwise. Thus it has a payoff function that is identical to a European binary call option with maturity $T_k$ and strike $q_k$, written on the cumulative emissions $x_t = \sum x^i_t$ of the whole economy. The dynamics of this non-tradable underlying is given by

$$dx_t = (y_t - \xi_t)dt + \sigma_x dW_t$$

in our framework, where $y_t = \sum y^i_t$ is the economy-wide business-as-usual emission rate following

$$dy_t = \mu_y(t)dt + \sigma_y(t)dZ_t$$

and $\xi_t = \sum \xi^i_t$ is economy-wide abatement. The parameter $\sigma_x$ as well as the functions $\mu_y(t)$ and $\sigma_y(t)$ result from aggregating the individual emissions processes (2), (3), and (4) such that $W_t$ and $Z_t$ are standard Wiener processes with uncorrelated increments. A distinctive feature of economy-wide emissions $x_t$ as a non-tradable underlying is that market participants can and obviously do influence its state through their abatement actions $\xi_t$. Since abatement reduces the companies’ greenhouse gas emissions, an emission permit is worth less than the corresponding strip of binary options in a scenario where no abatement is possible. The specific form of permit prices revealed by Proposition 2 obviously affects their precise distributional properties, but finds also expression in major structural features. First, there is a time-dependent upper bound for emission permit prices,

$$S_k(t) \leq \sum_{j=k}^{n} e^{-r(T_j-t)}p_j,$$

corresponding to a scenario of permit shortage for all compliance periods. In such a situation, one permit less means an additional penalty in the current and also in all following compliance periods due to the later delivery rule. This upper bound of emission permit prices obviously depends on the number of compliance periods in the system. Furthermore, the single value components of an emission permit are pulled to one of the two values — zero or the penalty
— by the end of the corresponding compliance period. This effect is weak as long as the period end is far and the uncertainty about penalties to accrue is high, such that medium values are attained with significant probabilities. As uncertainty decreases, the convergence to one of the two possible values becomes stronger. As a direct consequence, the prices of period-\( k \) and period-\( k + 1 \) permits are either identical at the end of period \( k \), or differ by the amount of the penalty, that is

\[
S_k(T_k) - S_{k+1}(T_k) = 1_{\{x_{T_k} > q_k\}}p_k.
\]

This means that there is a smooth transition when period-\( k \) permits are converted into period-\( k + 1 \) permits by banking if the economy is in permit surplus. In contrast, if the economy is short of permits, prices decrease by the penalty similar to the drop in value of a coupon bond after a coupon payment date.

Understanding an emission permit as a strip of binary options also sheds light on the volatility structure of permit prices. Given the dynamics (9) and (10) of cumulative emissions and the prevailing emission rate, we obtain local volatilities by applying Itô’s Lemma.

**Proposition 3 (Local Volatility).** Relative local volatility of emission permit prices, as given by

\[
\sigma_{S_k} = \frac{\sqrt{\left(\frac{\partial S_k}{\partial x} \sigma_x\right)^2 + \left(\frac{\partial S_k}{\partial y} \sigma_y\right)^2}}{S_k},
\]

is state- and time-dependent.

Due to the binary options characteristics of emission permits, the volatility of the single value components \( S_k - S_{k+1} \) clearly depends on the time to the end of the compliance period, \( T_k \), and is almost surely zero at the end of the period.\(^1\) The state-dependency becomes most evident when comparing scenarios of extremely high and medium emissions. In the first case, prices are close to the upper bound (11) and absolute volatility (that is, the numerator of (13)) is almost zero, since a marginal change in cumulative emissions or the prevailing emission rate hardly changes the probability of penalties. On the other hand, relative volatility is clearly higher in a scenario of medium emissions, since absolute volatility

\(^1\)Since \( \sigma_x > 0 \), Section 3.3 of Carmona et al. (2013) applies to our model, which implies that the cumulative emissions exactly hit the cap with zero probability, \( P_r \{x_{T_k} = q_k\} = 0 \).
is positive and prices are far away from the upper bound. For very low emissions, however, we do not get a clear indication for the volatility behavior, since absolute volatilities go to zero, but prices approach zero as well. We shed light on this aspect by considering the detailed volatility structure for a calibrated setting in Section 4.4.

4.2 Futures

It is market convention to consider the emission permits of the actual ongoing compliance period as spot permits, such that spot permits are period-1 permits until the end of compliance period 1, then period-2 permits until $T_2$, and so on. This view makes practical sense because only period-1 permits are usable for compliance in period 1, and they are converted into period-2 permits by banking at the end of the period. In accordance with that, permit futures deliver spot permits at maturity. While a futures contract with maturity $\bar{t} \in [0, T_1)$ delivers a period-1 permit, futures maturing at $\bar{t} \in (T_1, T_2)$ deliver period-2 permits. Based on this convention, we characterize the futures price curve in emission permit markets and discuss the applicability of classical convenience yield models.

Consider an *intra-period* futures contract at $t \in [0, T_1)$ with maturity $\bar{t} \in [t, T_1)$. Since period-1 permits cannot be used for compliance before $T_1$, they are pure investment assets before that date, and the storability of permits directly implies the standard cost-of-carry relation

$$F(t, \bar{t}) = e^{r(\bar{t} - t)} S_1(t),$$  \hspace{1cm} (14)

where $F(t, \bar{t})$ is the futures price. To the contrary, this does not apply to *inter-period* futures, i.e., the case $t \in [0, T_1)$, $\bar{t} \in (T_1, T_2)$. The futures contract delivers a period-2 permit, which can obviously not be used for compliance in period 1, while holding the spot (period-1) permits provides the option to use them at the end of period 1. This additional benefit can be quantified by the difference between period-1 and period-2 permit prices according to (8) together with (14):

$$F(t, \bar{t}) = e^{r(\bar{t} - t)} S_1(t) - e^{r(T_1 - t)} P \{ x_{T_1} > q_1 \} p_1.$$  \hspace{1cm} (15)

Consequently, the backwardation of inter-period futures defined as the difference between the current spot price and the discounted futures price, $B(t, \bar{t}) = S_1(t) - e^{-r(\bar{t} - t)} F(t, \bar{t})$, 16
is determined by the probability of permit shortage at the end of the ongoing compliance period. We summarize the implications of (14) and (15).

**Proposition 4** (Futures Price Curve). *The futures price curve has the following properties for all* \( t \in [0, T_1) \):

a) Futures are in contango within the compliance period, i.e., for \( t \in [t, T_1) \).

b) The backwardation of futures with maturity in the following compliance period, i.e., \( t \in (T_1, T_2) \), is given by

\[
B(t, \bar{t}) = e^{-r(T_1-t)} \mathbb{P}_t \{ x_{T_1} > q_1 \} p_1. \tag{16}
\]

In particular, inter-period futures are in contango if the probability of permit shortage at the end of the ongoing compliance period is 0, in weak backwardation if it is positive, and in strong backwardation if it is above \((e^{r(T_1-t)} - e^{-r(T_1-t)}) \frac{S_1(t)}{p_1}\).

This proposition provides a direct link to the classical commodity literature. Holding the spot asset endows the owner with an embedded usage and timing option as elaborated by Routledge et al. (2000) and Jarrow (2010), among others. However, while for classical commodities any point in time comes into question for exercising this usage option, emission permits can only be consumed at the end of a compliance period. Thus, this option is worthless within a compliance period and holding the spot permit has no advantage compared to futures maturing in the same period. To the contrary, the usage option becomes relevant if the futures’ maturity is in the next compliance period. If the economy is short of permits, the option is exercised to save penalty payments. Otherwise, it is not exercised for a number of leftover permits which are banked into the next period. Consequently, inter-period futures are in weak backwardation if the probability of permit shortage at the end of the compliance period is not exactly zero.

Since the seminal work of Brennan (1958), it is common to express the benefit of holding the spot asset rather than a futures contract as a convenience yield. In general, a time-dependent

\[\text{In line with Litzenberger and Rabinowitz (1995), futures with maturity } \bar{t} \text{ are in contango if } B(t, \bar{t}) \leq 0 \text{ and in (weak) backwardation if the futures price is lower than the compounded spot price, i.e., } B(t, \bar{t}) > 0. \text{ Strong backwardation means that the futures price is lower than the current spot price.} \]
stochastic instantaneous convenience yield $\delta_t$ changes the standard cost-of-carry relation to
\[
F(t, \bar{t}) = E_t \left\{ e^{r(\bar{t}-t)} - \int_t^{\bar{t}} \delta_s ds \right\} S_1(t). \tag{17}
\]

For given $t \in [0, T_1)$ we define $D_t(\bar{t}) := \int_t^{\bar{t}} \delta_s ds$ as the convenience yield from $t$ to $\bar{t}$. From (14) and (15) we can easily derive necessary properties of $D_t$.

**Proposition 5 (Convenience Yields).** The cumulative convenience yield $D_t$ fulfills

a) $D_t(\bar{t}) = 0$ for intra-period futures, i.e., $\bar{t} \in [t, T_1)$, and

b) $E_t \left\{ e^{-D_t(\bar{t})} \right\} = 1 - \frac{e^{-r(T_1-t)} P_t \{ x_{T_1} > q_1 \} P_1 S_1(t)}{S_t(\bar{t})}$ for inter-period futures, i.e., $\bar{t} \in (T_1, T_2)$.

It is obvious that $D_t$ has to “jump” in $T_1$ in order to fulfill the conditions of Proposition 5. Therefore it is not possible to define an instantaneous convenience yield $\delta_t$ in such way that $D_t$ has these properties. In particular, standard models like a mean-reverting stochastic convenience yield or a simple AR(4) process as used by Daskalakis et al. (2009) and Chevallier (2009) for inter-period futures do not satisfy Proposition 5. These models inevitably lead to relative mispricing when futures of two or more different maturities are considered.

### 4.3 Calculating Equilibrium Prices

We provide a strategy to calculate equilibrium permit prices as the solution of a system of partial differential equations (PDEs). Permit prices are determined by cumulative economy-wide emissions according to (8), but it is not simply possible to evaluate the expectation since this non-tradable underlying is influenced through endogenously derived abatement measures. More specifically, economy-wide abatement $\xi_t$ results as the sum of the companies’ abatement strategies that solve the individual optimization problems (5). Appendix B shows that we can simplify the problem by optimizing economy-wide abatement directly with respect to an aggregate problem.

**Proposition 6 (Global Problem).** Assume that economy-wide emissions $x_t$ still follow the dynamics (9), but let aggregate abatement $\xi_t$ be the solution of the global problem

\[
\min_{\xi} E_0 \left\{ \int_0^{T_n} e^{-rt} C(\xi_t) dt + \sum_{j=1}^n e^{-rT_j} p_j (x_{T_j} - q_j)^+ \right\}, \tag{18}
\]

where $C$ is the aggregate abatement cost function of the economy. Then, $S_1, \ldots, S_n$ defined by equation (8) are equilibrium permit price processes. Further, the permit price of the ongoing compliance period is equal to the instantaneous marginal abatement costs of the economy,

$$S_k(t) = \frac{\partial C}{\partial \xi}(\xi_t), \quad t \in [T_{k-1}, T_k].$$

(19)

In light of this result, we determine the optimal abatement strategy of the aggregate problem (18). We follow a backward induction approach, starting at the last compliance period $[T_{n-1}, T_n]$ and proceeding to the periods $[T_{n-2}, T_{n-1}], \ldots, [0, T_1]$. For each compliance period $k$, we include the period $k + 1$ solution into the terminal condition and settle the problem by dynamic programming. We state the resulting system of PDEs in here for a quadratic abatement cost function

$$C(\xi_t) = \frac{1}{2} \gamma \xi_t^2$$

(20)

with cost coefficient $\gamma$, and economy-wide business-as-usual emissions following an arithmetic Brownian motion

$$dy_t = \mu_y dt + \sigma_y dZ_t,$$

(21)

and refer to Appendix C for the derivation in the general case.

**Proposition 7 (PDEs).** For the global problem (18), optimal abatement $\xi_t$ at time $t \in [T_{k-1}, T_k]$ is given by

$$\xi_t = \frac{1}{\gamma} e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x}(t, x_t, y_t),$$

(22)

where $V_k$ is the time-$T_{k-1}$ expected value of an optimal strategy starting at $T_{k-1}$. $V_k$ solves the characteristic PDE

$$\frac{\partial V_k}{\partial t} = -y_t \frac{\partial V_k}{\partial x} + \frac{1}{2\gamma} e^{r(t-T_{k-1})} \left( \frac{\partial V_k}{\partial x} \right)^2 - \frac{\partial V_k}{\partial y} \mu_y - \frac{1}{2} \frac{\partial^2 V_k}{\partial x^2} \sigma_x^2 - \frac{1}{2} \frac{\partial^2 V_k}{\partial y^2} \sigma_y^2$$

(23)

with boundary condition

$$V_k(T_k, x_{T_k}, y_{T_k}) = e^{-r(T_k-T_{k-1})} (p_k(x_{T_k} - q_k)^+ + V_{k+1}(T_k, x_{T_k}, y_{T_k}))$$

(24)

and $V_{n+1} = 0$. 19
As shown by Proposition 6, the solution for optimal economy-wide abatement at time $t$ directly implies the equilibrium permit price of the ongoing compliance period through (19). Thus, permit prices can be computed by numerically solving the system of PDEs (23), (24), starting from period $n$ and proceeding backwards.

4.4 Calibration

We illustrate the specific features of emission permit prices and volatilities as well as the role of abatement actions and subsequent compliance periods within a calibrated setting in line with Phase II of the EU Emissions Trading System. Model parameters are chosen as stated by Table 1. Phase II from 2008 to 2012 is followed by compliance periods of eight years length from 2013 onwards. Figures on the number of permits to be allocated for each of the compliance periods are reported by the European Commission.\(^{16}\) Penalties are 100 Euros per ton of exceeding emissions in Phase II and increase “in accordance with the European index of consumer prices”,\(^{17}\) which we assume as 2.5% per year. The parameters for the economy-wide emissions processes (9) and (10) are extracted from yearly emissions data provided by the European Environment Agency (EEA).\(^{18}\) This data covers carbon dioxide equivalent emissions of all industry sectors except land use, land-use change and forestry (LULUCF) activities in the 27 EU member states and is considered as a good proxy for emissions within the EU ETS (Ellerman and Buchner 2008). We use data from 1995 to 2004 that is not biased by the introduction of the EU ETS. Further, the coefficient of the abatement cost function (20) is in the region of 0.1 to 0.2, as suggested by a linear interpolation of Europe’s marginal abatement cost curve (Nauclér and Enkvist 2009; Cline 2011). Accordingly we consider two different scenarios, $\gamma_1 = 0.1$ and $\gamma_2 = 0.2$, which allows us to analyze the impact of abatement measures. Finally, we set the constant risk-free interest rate $r$ to 4%.

Figure 1 depicts permit prices $S_1$, calculated according to Proposition 7, dependent on economy-wide realized emissions $x_t$ and the prevailing business-as-usual emission rate $y_t$ for three different points in time at the beginning, halfway through, and right before the end of the compliance period $[0, T_1]$. We compare the two abatement cost scenarios and contrast

\(^{16}\)The amount of allocated permit per year declines in a linear manner according to MEMO/08/796 of the European Commission.


\(^{18}\)See \url{http://www.eea.europa.eu/}. 
Table 1: Summary of parameter values for a setting in accordance with Phase II of the EU ETS. The allocation and penalties for four compliance periods from 2008 to 2012, 2013 to 2020, 2021 to 2028, and 2029 to 2036 are chosen as outlined by the European Parliament (see MEMO/08/796). Parameters for the emissions processes (9) and (21) are inferred from data provided by the European Environment Agency (EEA). We consider a scenario of low abatement costs (with coefficient $\gamma_1$) compared to the case of higher costs ($\gamma_2$). $r$ is the constant risk-free interest rate.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Compliance Period $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>End of compliance period</td>
<td>$T_k$</td>
</tr>
<tr>
<td>Allocation (Million permits)</td>
<td>$e_k$</td>
</tr>
<tr>
<td>Penalty (Euro)</td>
<td>$p_k$</td>
</tr>
<tr>
<td>Drift of emission rate</td>
<td>$\mu_y$</td>
</tr>
<tr>
<td>Volatility of emission rate</td>
<td>$\sigma_y$</td>
</tr>
<tr>
<td>Volatility of emission shocks</td>
<td>$\sigma_{\varepsilon}$</td>
</tr>
<tr>
<td>Abatement cost coefficient (low)</td>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>Abatement cost coefficient (high)</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>Interest rate (p.a.)</td>
<td>$r$</td>
</tr>
</tbody>
</table>

The full four-period setting to a setting that takes only the first two compliance periods into account. Figure 2 shows volatilities $\sigma_{S_1}$ for the low abatement cost scenario. Since the level and shape of the volatility surface for high abatement costs is very similar, we refrain from displaying it.

The plots elucidate the strongly nonlinear and time-variant dependency of emission permit prices on business-as-usual emissions. Especially towards the end of the compliance period, the binary options character highlighted by Proposition 2 becomes clearly visible. These option characteristics are also responsible for the state- and time-dependent nature of local volatilities according to Proposition 3. Particularly, we have established that volatility goes to zero for high permit prices resulting from very high levels of realized or prevailing emissions. Figure 2 reveals in addition that volatility is highest for very low emissions. At the end of the compliance period, the binary component of permit prices shows up in the volatility surface, and relative volatility is much higher for low prices corresponding to non-penalty states. Such a negative relation between prices and volatilities is known from equity markets as the 'leverage effect' (Black 1976b; Christie 1982). The technical analogy to emission
permits is worth noting: Just like stocks are options on the firm value, emission permits are (strips of binary) options on economy-wide emissions. The resulting distributional properties explain the ‘leverage effect’ on equity markets (Geske 1979) and a similar effect for emission permits. A difference is that the payoff structure for emission permits as a strip of binary options is much more complex than for a single option and the resulting permit price distributions can be various when some of these binary options are in- and some are out-of-the-money. Lower permit prices lead to higher permit price volatilities for the most part, but the relation is not necessarily monotonous for medium emissions scenarios, as the plots for the four-period setting show.

Moreover, our analysis reveals the role of additional consecutive compliance periods for permit prices and volatilities. Cap-and-trade systems equipped with more compliance periods naturally result in a higher value of today’s emission permits. The increase of today’s price illustrates to what extent a regulator can incentivize additional abatement by announcing further consecutive compliance periods. Additionally, a higher number of compliance periods leads to lower volatilities due to two effects. First, the price components attributable to the compliance periods in the remote future react less sensitively to changes in today’s cumulative emissions or the prevailing emission rate. Second, the tightening allocation in later compliance leads to higher values — and thus lower relative volatilities — for these value components, such that the value components coming from additional compliance periods weaken the high relative volatility from the current period and cause a smoothing effect. This is desired by policy makers because stable prices increase the confidence to the trading system and tend to trigger early investment into environment-friendly technologies. Overall, the considerable impact on both prices and volatilities makes the choice of number and length of consecutive compliance periods an important design element.

The effects of abatement are indicated by the difference between emission permit prices in the low and the high abatement cost scenario. Lower abatement costs lead to increased abatement activities resulting in lower permit prices according to the equilibrium mechanism of the emissions market. Moreover, the possibility of abatement dampens the sensitivity of emission permit prices towards changes in business-as-usual emissions. While increasing business-as-usual emissions lead to higher abatements, abatement is small for low business-as-usual emission levels, imposing an additional nonlinearity to the relationship between business-as-usual emissions and resulting permit prices. As a consequence, models not accounting
Figure 1: Permit price $S_1$ dependent on cumulative realized emissions $x_t$ and the prevailing business-as-usual emission rate $y_t$. We consider a two-period and a four-period setting in line with the EU ETS and consider in each case three different time points within the first compliance period: one and a half ($t = 1.5$) and three years ($t = 3$) after the compliance period’s start and shortly before the period’s end ($t = 4.9$). The green plots represent the high abatement cost scenario, the blue plots the case of low abatement costs. Parameter values are chosen according to Table 1.
Figure 2: Relative local volatility $\sigma_{S_1}$ according to (13), dependent on cumulative realized emissions $x_t$ and the prevailing business-as-usual emission rate $y_t$. We consider a two-period and a four-period setting in line with the EU ETS and consider in each case three different time points within the first compliance period: one and a half ($t = 1.5$) and three years ($t = 3$) after the compliance period’s start and shortly before the period’s end ($t = 4.9$). Parameter values are chosen according to Table 1.
for abatement measures may potentially misestimate future permit price distributions. In particular, a model not incorporating abatement calibrated in a regime of medium emission levels predicts too high emissions and thus too high prices for a future regime of increased business-as-usual emissions, and too low prices when business-as-usual emissions fall.

4.5 Option Pricing and Volatility Smiles

Options on emission permits are important instruments for polluters to manage nonlinear exposures to permit price risk, for example due to investments into environment-friendly production technologies that are operated only when permit prices are above a certain threshold. In fact, options account for about 10% of the overall transaction volume related to the EU ETS in 2011.\(^{19}\) The volatility smile of option prices aggregates structural pricing patterns in a comprehensive way and typically has a characteristic shape for different types of underlyings. While the smile is predominantly downward-sloping for equity options, an upward-sloping smile is typical for commodity markets.\(^{20}\) The shape of the volatility smile results from both the (state-dependent) risk premia of market participants and the distributional properties of the underlying prices under the real measure. Our risk-neutral model framework is well-suited to analyze how the distributional properties of emission permit prices originating from the design of today’s emission trading systems translate to the volatility smile in permit markets.

Since emission permit options are written on a strip of European binary call options according to Proposition 2, the pricing problem is structurally similar to compound options first studied by Geske (1979). As revealed in the last section, local volatilities of emission permits are in general negatively related to prices, suggesting a downward-sloping smile in this market. The shape of the smile is, however, not immediately implied by local volatilities since it is also affected by the time-dependent volatility behavior until the option’s expiry.

We therefore perform an extensive simulation study to characterize the volatility smile within our calibrated model for a number of different time points, emissions scenarios, and option maturities. In our risk-neutral setting the price of a European call option with strike \(K\)

\(^{19}\)See Kossoy and Guigon (2012).

\(^{20}\)We refer to Ederington and Guan (2013) for an extensive analysis of the volatility smiles for a number of different markets.
Figure 3: Slope of implied volatility smiles for different emissions scenarios $x_t, y_t$ and time parameters $t, \bar{t}$. Each of the 25 squares in each plot corresponds to one emissions scenario, and the color indicates the slope of the smile according to the given legend. The slope is calculated based on implied volatilities $IV^- K$ and $IV^+ K$ at strike prices $K^-$ and $K^+$, with $K^- < F(t, \bar{t}) < K^+$, as $\frac{IV^+ K - IV^- K}{K^+ - K^-}$.

and maturity $\bar{t}$ written on emission permit futures with the same maturity\textsuperscript{21} equals the discounted expected payoff, that is

$$C(t, \bar{t}, K) = e^{-r(\bar{t}-t)} \mathbb{E}_t \left\{ (F(\bar{t}, \bar{t}) - K)^+ \right\}.$$  \hspace{1cm} (25)

Given $t, \bar{t}, x_t$, and $y_t$, we calculate call option prices (25) by Monte-Carlo simulation and compute the related Black (1976a) implied volatility $IV_{ATM}$ at-the-money, i.e., for strike $K = F(t, \bar{t})$, as well as for one strike price above and one below the current futures price, given by $K^\pm = F(t, \bar{t})(1 \pm IV_{ATM} \sqrt{t-\bar{t}})$.

We capture the slope of the volatility smile by $\frac{IV^+ K - IV^- K}{K^+ - K^-}$, such that positive (negative) values stand for an upward-(downward-)sloping volatility smile, and values close to zero

\textsuperscript{21}As for classical commodities, the underlying is usually a futures contract rather than an emission permit itself. We abstract from the fact that there are usually a few days between the option’s expiry and the maturity date of the futures contract.
imply that the smile is almost symmetric. For the sake of brevity we report results only for the setting of two compliance periods and low abatement costs, but we obtain qualitatively similar results for all other cases. Figure 3 illustrates the slope of the volatility smile for the different emissions scenarios and time parameters. It is eye-catching that the smile is downward-sloping or almost symmetric for the vast majority of emissions scenarios, while there are no cases with a clearly upward-sloping smile. In fact, the slope is negative for 141 of the 150 scenarios. The strongest upward-slope is 0.002 which is — compared to the strongest downward-slope of -0.0166 — very close to a symmetric smile. Further, the downward-slope of the volatility smile is strongest for scenarios of very low emissions. These results accord to the negative relation of emission permit prices and volatilities and reveal how this pattern translates to option prices.

Recall that the volatility smile shapes in our study are induced by the characteristics of permit price distributions under the real measure. As for traditional energy commodities, it can be assumed that market participants perceive high permit prices as bad states (Geman 2005). Therefore the inclusion of risk premia might reduce the downward-slope of the smile predicted by the distribution of underlying permit prices in our model.

5 Empirical Evidence

We analyze the predictions of our model in light of empirical evidence from existing emission trading systems. The second compliance period (Phase II) of the EU ETS from 2008 to 2012 endows us with a rich amount of data on European Union Allowance (EUA) spot, futures, and option prices traded on the European Climate Exchange (ECX). Additionally, the auctions of the US Acid Rain Program for SO\textsubscript{2} emissions from 1996 to 2011 provide a unique data sample to verify our predictions on the backwardation of inter-period futures. Proposition 3 highlights the state- and time-dependency of permit price volatilities, and the calibration in Section 4.4 particularly reveals that volatilities are high when prices are low and vice versa. We evaluate this prediction by analyzing the time series of daily prices and

\textsuperscript{22}Daily data is provided directly by the ECX, while we obtain intraday data from Thomson Reuters. We thank SIRCA for providing access to the Thomson Reuters DataScope Tickhistory archive, \url{http://www.sirca.org.au}.

\textsuperscript{23}SO\textsubscript{2} auction data is published by the US Environmental Protection Agency (EPA), see \url{http://www.epa.gov}.
volatilities of EUAs from 2008 to 2012. To calculate the volatility $\hat{\sigma}(t)$ on day $t$, we employ a model-free approach based on intraday price data given by

$$\hat{\sigma}(t) = \sqrt{\frac{252}{N} \sum_{i=1}^{N} R_{t,i}^2},$$  \hspace{1cm} (26)$$

where $R_{t,i}$, $i = 1, \ldots, N$ are 5-minute intraday log returns and we annualize according to a number of 252 trading days per year. Since intraday EUA spot data is available only limitedly due to partially thin trading, we base our analysis on EUA futures maturing before the end of Phase II instead. Prices and volatilities of intra-period futures and spot permits can be considered equivalently due to the standard cost-of-carry relation within a compliance period.\(^{25}\) We construct a time series of intra-period futures prices and volatilities from

\(^{24}\)As Andersen et al. (2001) point out, the model-free approach based on intraday data reflects the true volatility of returns much more accurately than measures based on lower frequencies. Close-to-open returns are not included into the sum.

\(^{25}\)The standard cost-of-carry relationship within a compliance period is predicted by our model, see Proposition 4. Since this result is empirically confirmed by the existing literature, see Uhrig-Homburg and Wagner
January 2008 to June 2012 by taking the EUA December futures next to maturity and rolling over to the next contract on the last day of October.

The time-varying nature of permit price volatilities is clearly observable in Figure 4. Volatilities increase from the beginning of the sample until the first quarter of 2009, before declining to a comparably low level in 2010 and the first half of 2011. In the second half of 2011 and 2012, daily volatilities are significantly higher again. A comparison with the price level of EUA futures clearly reveals that higher volatilities generally correspond to lower prices. Prices are lowest in the beginning of 2009 and in 2012, when volatilities reach their highest level. Overall, the sample correlation of daily prices and volatilities is -0.365, which clearly confirms the prediction.

Let us further analyze the futures price curve in emission permit markets in view of our model results. Proposition 4 states that the backwardation of futures maturing in the following compliance period increases in the expected probability of permit shortage. In particular, inter-period futures should be strongly backwardated in case of a severe permit scarcity in the ongoing compliance period, while they should be almost in normal contango in case of a large over-allocation. For Phase II of the EU ETS, market reports clearly indicate a considerable permit surplus that was realized from 2011 on. Thus, the backwardation of EUA futures maturing in December 2013 and 2014 should be small according to the predictions of our model. As illustrated by Figure 5 for three exemplarily selected days, the futures price curve virtually was in normal contango also across the compliance periods during the years 2011 and 2012. Although the correspondence of low emissions expectations to the lack of backwardation in the EU ETS is in line with our predictions for one possible scenario, the evidence could be strengthened by analyzing a scenario of high expected emissions as well. Unfortunately, EUA futures maturing in December 2013 were not liquidly traded before 2011, such that inter-period futures data for a scenario with higher expected emissions is not available for the EU ETS.

Therefore we additionally investigate the yearly auction results of the US Acid Rain Program for sulfur dioxide (SO$_2$) emissions, which provide a unique sample to analyze the backwardation across compliance periods dependent on expected emissions. Although the SO$_2$ market is extensively studied in the economic literature (e.g., Joskow et al. 1998; Schmalensee et al. 1998), this aspect is not covered by the existing research. The basic rules of the SO$_2$ emission

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(2009) and Rittler (2012), we refrain from an own empirical evaluation in here.
trading system are very similar to those of the EU ETS, and thus in line with our model assumptions. In our terminology, every single year is a compliance period in the SO$_2$ system since borrowing is prohibited between different years. The yearly SO$_2$ permit auction provides prices for spot permits for the ongoing year as well as seven-year advance permits to be used not earlier than seven years in the future. We further obtain data on the overall amount of left-over permits at the end of each year that market participants banked into the next year. Assuming rational expectations, this data proxies the ex-ante expectation of emissions, such that a low (high) number of left-over permits also indicates low (high) expected emissions for that year. Overall we obtain 16 observations from 1996 to 2011 to verify our predictions, as illustrated by Figure 6. In neat accordance with Proposition 4, the backwardation in the SO$_2$ market is strongest in the years 2005 and 2006 when the permit surplus is on the lowest level. In these cases, the ex-ante probability of a permit shortage at the end of the ongoing compliance period is relatively high, and the current emission permits can be used to avoid penalty payments. This is not possible, however, with seven-year advance permits, which leads to the backwardation. In the years before 2005, it is clearly observable how an advancing scarcity of permits leads to a higher backwardation. In the same way, an increasing over-allocation after 2006 reduces the observable backwardation.
more and more.

The application of our model to option pricing in Section 4.5 suggests that the negative relation of permit prices and volatilities finds expression in a downward-sloping volatility smile in emission permit markets. We analyze this result based on prices of European options written on EUA futures\(^{26}\) and illustrate the average volatility smiles for the years 2010, 2011, and 2012 in Figure 4. Confirming the model prediction, the average smile is clearly downward-sloping for each of the three years. Moreover, the decrease of EUA prices from 2010 to 2012 is accompanied by an increasing downward-slope of the volatility smile. This observation corresponds to the results of our simulation analysis, which predicts the strongest downward-slopes for scenarios of very low realized and prevailing emissions.

Overall, we conclude that all main testable predictions of our model are confirmed by empirical data from emission permit markets. It can be seen as a natural next step to conduct a more rigorous analysis of our model by evaluating its fit to empirical emission permit prices and related derivatives in comparison to classical standard models for asset price dynam-

\(^{26}\)We clean the sample according to standard criteria widely used in the option pricing literature, see for example Bakshi et al. (1997). In particular, we exclude options traded at prices smaller than 0.06 Euro as well as observations of call or put prices violating the no-arbitrage condition \(C(t, T, K) \geq e^{-r(T-t)}(F(t, T) - K)\) or \(P(t, T, K) \geq e^{-r(T-t)}(K - F(t, T))\).
6 Conclusion

We develop a stochastic equilibrium model for emission trading systems in order to analyze the characteristic properties of emission permits as a novel financial instrument. Accounting for the specific design of today’s state-of-the-art emission trading systems, we show that an emission permit can be characterized as a strip of European binary options written on economy-wide emissions. This option characteristics gives rise to several general properties of emission permit prices, especially a state- and time-dependent volatility structure. Since companies reduce their emissions by implementing abatement measures, an emission permit is worth less than the corresponding strip of binary options in a model that ignores abatement opportunities. Our model further reveals the hybrid nature of emission permits between investment and consumption assets, which leads to a futures price curve that is in contango within compliance periods, but backwardated across them. We calibrate our model to a setting in line with the EU ETS and study the valuation of European options in this market. Empirical evidence from existing emission trading systems shows that our model is able to
capture the stylized facts of emission permit prices and related derivatives.
A Optimal Conditions for Individual Companies

We characterize the optimal trading and abatement strategies \((\theta^i, \xi^i)\) of the individual companies for given permit price processes \(S_1(t), \ldots, S_n(t)\). The resulting optimality conditions relate the company’s marginal abatement costs and the expected penalty payments to the given permit prices. We first decompose the individual optimization problem (5) into a recursive system of \(n\) simpler problems, one for each compliance period of the emission trading system, including the value function \(V^i_k\) of the period \(k\) problem into the terminal condition of the period \(k-1\) problem, with \(V^i_{n+1} = 0\):

\[
V^i_k(t, x^i_t, y^i_t, Q^i_{k,t}, \ldots, Q^i_{n,t}) = \min_{(\theta^i_k, \xi^i_k)} \mathbb{E}_{T_{k-1}} \left\{ \int_t^{T_k} e^{-r(s-T_k-1)} C^i(\xi^i_s) ds + \sum_{j=k}^n \int_t^{T_k} e^{-r(s-T_k-1)} S^i_j(s) \theta^i_j ds + e^{-r(T_k-T_{k-1})} \left( p_k(x^i_k - Q^i_{k,T_k})^+ + V^i_{k+1}(T_k, x^i_{T_k}, y^i_{T_k}, Q^i_{k+1,T_k}, \ldots, Q^i_{n,T_k}) \right) \right\},
\]

(27)

for \(t \in [T_{k-1}, T_k]\) and \(k = 1, \ldots, n\), where \((\theta^i_k, \xi^i_k)\) is the restriction of \((\theta^i, \xi^i)\) to the time interval \([T_{k-1}, T_k]\) and we introduce the additional state variables \(Q^i_{k,t} = \sum_{j=1}^k \left( \mathbb{E}_j^i + \int_0^{\min(t,T_j)} \theta^i_j ds \right)\) in generalization of \(Q^i_k = Q^i_{k,T_k}\). According to the dynamic programming principle (see Bertsekas 1976), an optimal solution \((\theta^i, \xi^i)\) of the original problem is also a solution of the decomposed problem (27), and \(V^i_1\) is identical to the value function of the original problem for \(t \in [0, T_1]\). The dynamics of the state variables follow from (4), (2), and the definition of \(Q^i_{k,t}\) as

\[
\begin{align*}
dx^i_t &= (y^i_t - \xi^i_t) dt + \sigma^i_t dW^i_t, \\
dy^i_t &= \mu^i_y(t) dt + \sigma^i_y(t) dZ^i_t, \\
dQ^i_{l,t} &= \sum_{j=k}^l \theta^i_{l,t} dt, \quad l = k, \ldots, n.
\end{align*}
\]

(28)

We derive optimality conditions for the trading and abatement strategy \((\theta^i, \xi^i)\) by applying the stochastic maximum principle to the problems (27), proceeding recursively from \(k = n\).
to $k = 1$. A strategy $(\theta^i_n, \xi^i_n)$ for period $n$ that minimizes the costs according to (27) maximizes the Hamiltonian

$$H_n(t, x^i, y^i, \theta^i, \xi^i, \rho_n) = \rho_{n,x^i}(t) \cdot (y^i_t - \xi^i_t) + \rho_{n,y^i}(t) \cdot \mu^i_y(t) + \rho_{n,Q^i_n}(t) \cdot \theta^i_{n,t} - e^{-r(t-T_{n-1})}(C^i(\xi^i_t) + S_n(t)\theta^i_{n,t}).$$

at every point in time $t \in [T_{n-1}, T_n]$, where $(\rho_{n,x^i}, \rho_{n,y^i}, \rho_{n,Q^i_n})$ are the adjoint processes corresponding to the state variables $(x^i, y^i, Q^i_n)$. Differentiating (29) with respect to the control variables and setting the derivatives to zero yields the optimality conditions

$$\frac{\partial H_n}{\partial \xi^i_n} = -\rho_{n,x^i}(t) - e^{-r(t-T_{n-1})}\frac{\partial C^i}{\partial \xi^i_t} = 0,$$

$$\frac{\partial H_n}{\partial \theta^i_n} = \rho_{n,Q^i_n}(t) - e^{-r(t-T_{n-1})}S_n(t) = 0.$$

It remains to derive the adjoint processes $\rho_{n,x^i}$ and $\rho_{n,Q^i_n}$, which are defined by the stochastic differential equations

$$d\rho_{n,x^i}(t) = \omega_{n,x^i}(t)dW^i_t + \zeta_{n,x^i}(t)dZ^i_t,$$

$$d\rho_{n,Q^i_n}(t) = \omega_{n,Q^i_n}(t)dW^i_t + \zeta_{n,Q^i_n}(t)dZ^i_t,$$

with stochastic processes $(\omega_{n,x^i}, \omega_{n,Q^i_n}, \zeta_{n,x^i}, \zeta_{n,Q^i_n})$ and terminal conditions

$$\rho_{n,x^i}(T_n) = -e^{-r(T_n-T_{n-1})}1_{\{x^i_{T_n} > Q^i_{n,T_n}\}}p_n,$$

$$\rho_{n,Q^i_n}(T_n) = e^{-r(T_n-T_{n-1})}1_{\{x^i_{T_n} > Q^i_{n,T_n}\}}p_n.$$

We can directly identify the solution

$$\rho_{n,x^i}(t) = -e^{-r(T_n-T_{n-1})}P_t \{x^i_{T_n} > Q^i_{n,T_n}\} p_n,$$

$$\rho_{n,Q^i_n}(t) = e^{-r(T_n-T_{n-1})}P_t \{x^i_{T_n} > Q^i_{n,T_n}\} p_n.$$

---

27 See Yong and Zhou (1999), Chapter 3 for a comprehensive introduction of the stochastic maximum principle for optimal control problems. For our problem, we apply the stochastic maximum principle for the case of a non-smooth terminal condition, see Chighoub et al. (2009).

28 For the existence of a regular solution we refer to Carmona et al. (2013).
The adjoint processes can be interpreted as the shadow price of the corresponding state variable. For example, $\rho_{n,Q}$ is the value that can be attributed to having one marginal unit of period-$n$ permits more. Here, this is the discounted penalty weighted by the probability of penalties to accrue, which makes perfect economic sense.

Inserting the adjoint processes into (30) we arrive at the condition

\[
\frac{\partial C_i}{\partial \xi_i}(\xi_i t) = e^{-r(T_n-t)}P_t \{ x_{T_n}^i > Q_{n,T_n}^i \} p_n = S_n(t),
\]

which proves Proposition 1 for $t \in [T_{n-1}, T_n]$ and $k = n$.

Before proceeding to $k = n - 1$, note that negative of the adjoint process for a state variable equals the first derivative of the value function with respect to the same variable (see Clarke and Vinter 1987), that is

\[
-\rho_{n,x^i}(t) = \frac{\partial V_i}{\partial x^i}(t), \quad -\rho_{n,y^i}(t) = \frac{\partial V_i}{\partial y^i}(t), \quad -\rho_{n,Q_n}(t) = \frac{\partial V_i}{\partial Q_n}(t)
\]

for $t \in [T_{n-1}, T_n]$.

Now consider the optimal control problem (27) for $k = n - 1$. In this case the Hamiltonian is given by

\[
H_{n-1}(t, x^i, y^i, \theta^i, \xi^i, \rho_{n-1}) = \rho_{n-1,x^i}(t) \cdot (y^i_t - \xi^i_t) + \rho_{n-1,y^i}(t) \cdot \mu^i_y(t)
\]

\[+ \rho_{n-1,Q^i_{n-1}}(t) \cdot \theta^i_{n-1,t} + \rho_{n-1,Q_n}(t) \cdot (\theta^i_{n-1,t} + \theta^i_{n,t})
\]

\[- e^{-r(t-T_{n-2})} (C_i(\xi^i_t) + S_{n-1}(t)\theta^i_{n-1,t} + S_n(t)\theta^i_{n,t}),
\]

and as before we obtain the optimum by differentiating with respect to the control variables and setting the derivatives to zero:

\[
\frac{\partial H_{n-1}}{\partial \xi_i}(\xi_i t) = -\rho_{n-1,x^i}(t) - e^{-r(t-T_{n-2})} \frac{\partial C_i}{\partial \xi_i}(\xi_i t) = 0,
\]

\[
\frac{\partial H_{n-1}}{\partial \theta^i_{n-1}} = \rho_{n-1,Q^i_{n-1}}(t) + \rho_{n-1,Q_n}(t) - e^{-r(t-T_{n-2})}S_{n-1}(t) = 0,
\]

\[
\frac{\partial H_{n-1}}{\partial \theta^i_n} = \rho_{n-1,Q_n}(t) - e^{-r(t-T_{n-2})}S_n(t) = 0.
\]
It is left to insert the adjoint processes \( (\rho_{n-1,x^i}, \rho_{n-1,Q^i_{n-1}}, \rho_{n-1,Q^n_i} ) \) solving the equations

\[
\begin{align*}
    d\rho_{n-1,x^i}(t) &= \omega_{n-1,x^i}(t) dW^i_t + \zeta_{n-1,x^i}(t) dZ^i_t, \\
    d\rho_{n-1,Q^i_{n-1}}(t) &= \omega_{n-1,Q^i_{n-1}}(t) dW^i_t + \zeta_{n-1,Q^i_{n-1}}(t) dZ^i_t, \\
    d\rho_{n-1,Q^n_i}(t) &= \omega_{n-1,Q^n_i}(t) dW^i_t + \zeta_{n-1,Q^n_i}(t) dZ^i_t,
\end{align*}
\]  

(38) with terminal conditions

\[
\begin{align*}
    \rho_{n-1,x^i}(T_{n-1}) &= -e^{-r(T_{n-1} - T_{n-2})} (1_{\{x^i_{T_{n-1}} > Q^i_{n-1,T_{n-1}}\}} p_{n-1} \\
    &\quad + \frac{\partial V^n_i}{\partial x^i}(T_{n-1}, x^i_{T_{n-1}}, y^i_{T_{n-1}}, Q^i_{n,T_{n-1}})), \\
    \rho_{n-1,Q^i_{n-1}}(T_{n-1}) &= e^{-r(T_{n-1} - T_{n-2})} 1_{\{x^i_{T_{n-1}} > Q^i_{n-1,T_{n-1}}\}} p_{n-1}, \\
    \rho_{n-1,Q^n_i}(T_{n-1}) &= -e^{-r(T_{n-1} - T_{n-2})} \frac{\partial V^n_i}{\partial Q^n_i}(T_{n-1}, x^i_{T_{n-1}}, y^i_{T_{n-1}}, Q^i_{n,T_{n-1}}),
\end{align*}
\]  

(39)

After inserting the derivatives of \( V^n_i \) according to (33) and (35), we identify the solution

\[
\begin{align*}
    \rho_{n-1,x^i}(t) &= -\sum_{j=n-1}^n e^{-r(T_{j,T_{n-2}}-T_{j-1,T_{n-2}})} P_t \{ x^i_{T_j} > Q^i_{j,T_j} \} p_j, \\
    \rho_{n-1,Q^i_{n-1}}(t) &= e^{-r(T_{n-1} - T_{n-2})} P_t \{ x^i_{T_{n-1}} > Q^i_{n-1,T_{n-1}} \} p_{n-1}, \\
    \rho_{n-1,Q^n_i}(t) &= e^{-r(T_{n-2} - T_{n-1})} P_t \{ x^i_{T_{n}} > Q^n_i \} p_{n}.
\end{align*}
\]  

(40)

Using the adjoint processes in (37) finally yields the optimality conditions

\[
\begin{align*}
    \frac{\partial C^n_i}{\partial \xi^i}(t) &= \sum_{j=n-1}^n e^{-r(T_{j,T_{n-2}})} P_t \{ x^i_{T_j} > Q^i_{j,T_j} \} P_{j} = S_{n-1}(t), \\
    e^{-r(T_{n-2})} P_t \{ x^i_{T_{n}} > Q^n_i \} p_n = S_{n}(t),
\end{align*}
\]  

(41)

which proves Proposition 1 for \( t \in [T_{n-2}, T_{n-1}] \) and \( k = n - 1 \). Proceeding along the same lines for \( k = n - 2 \) to \( k = 1 \) completes the proof of Proposition 1.
B Individual Optimality of Global Optimal Solution

We construct equilibrium permit price processes $S_1, \ldots, S_n$ through equation (8) by letting the individual abatement strategies $\xi^i$ driving the economy-wide realized emissions $x_t$ be given by the solution of the joint cost problem of the whole economy $I$. Then we simplify the joint cost problem to the global problem (18) acting on aggregate volumes. The joint cost problem is to minimize the sum of the individual companies' costs according to (5) by an optimal trading and abatement strategy $(\Theta, \Xi) = (\theta^i, \xi^i)_{i \in I}$ subject to the market-clearing constraint. Since costs and revenues from individual trading cancel out on aggregate under market clearing, we obtain the problem

$$
\min_{(\Theta, \Xi)} E_0 \left\{ \sum_{i \in I} \left( \int_0^T e^{-rt} C^i(\xi^i_t) dt + \sum_{j=1}^n e^{-rT_j} p_j (x^i_{T_j} - Q^i_j)^+ \right) \right\}, \quad (42)
$$

Since in (42) the trading strategy $\Theta$ is only relevant for the penalty payments (by entering $Q^i_k$), we directly observe that a market-clearing trading strategy $\Theta$ optimizes (42) if and only if $1\{x^i_{T_k} > Q^i_k\} = 1\{x^i_{T_k} > q_k\}$ for $k = 1, \ldots, n$. On the other hand, the abatement strategy $\Xi$ enters both the penalties (through $x^i_{T_k}$) and the abatement costs. Applying the stochastic maximum principle along the lines of Appendix A yields that an optimal abatement strategy $\Xi$ fulfills

$$
\frac{\partial C^i}{\partial \xi^i}(\xi^i_t) = \sum_{j=k}^n e^{-r(T_j-t)} P_t \left\{ x^i_{T_j} > q_j \right\} p_j, \quad t \in [T_{k-1}, T_k], \quad (43)
$$

for all companies $i \in I$.

Now assume that an optimal abatement strategy of (42) is given by $(\Theta^*, \Xi^*)$ and choose

---

29 Our approach partly builds on Seifert et al. (2008) and Carmona et al. (2009). In the appendix of Seifert et al. (2008) it is shown that the global problem acting on aggregate volumes is equivalent to the sum of all companies' individual solutions, under the crucial assumption that all companies' emissions are driven by the same Wiener process. By a more general approach, Carmona et al. (2009) prove that the solution of the global problem is optimal for the individual problems also without this assumption.

30 That means, if the number of period-$k$ permits in the whole economy is not sufficient to cover economy-wide emissions at $T_k$, companies distribute the available permits in such way that none of the companies individually has left-over permits. On the other hand, in case of a permit surplus companies distribute the permits in such way that none of them has to pay penalties.
permit price processes $S^*_k$ by

$$S^*_k(t) = \sum_{j=k}^{n} e^{-r(T_j - t)} \mathbb{P}_t \left\{ x^*_i > q_j \right\} p_j, \quad t \in [0, T_k],$$

(44)
in line with (8), where the asterisks indicate that economy-wide realized emissions $x^*_i$ follow the dynamics (9) with abatements $\xi_t = \sum_{i \in I} \xi^i_t$ chosen according to the optimal strategy $\Xi^*$ of the global problem (42).

We show that given the permit price processes (44), the optimal strategy $(\Theta^*, \Xi^*)$ of the joint cost problem (42) is also optimal for the individual problems (5), implying that $S^*_1, \ldots, S^*_n$ are equilibrium permit prices. For that, expand the expected value in (5) by adding and subtracting the term $\sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt$ and split it into two parts according to

$$E_0 \left\{ \int_0^{T_n} e^{-rt} C^i(\xi^i_t) dt + \sum_{j=1}^{n} \int_0^{T_j} e^{-rt} S^*_j(t) \theta^i_{j,t} dt + \sum_{j=1}^{n} e^{-rT_j} p_j (x^i_{T_j} - Q^i_j)^+ \right\}$$

$$= E_0 \left\{ \int_0^{T_n} e^{-rt} C^i(\xi^i_t) dt - \sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt \right\}$$

$$+ E_0 \left\{ \sum_{j=1}^{n} \left( \int_0^{T_j} e^{-rt} S^*_j(t) \theta^i_{j,t} dt + \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt + e^{-rT_j} p_j (x^i_{T_j} - Q^i_j)^+ \right) \right\}.$$

(45)

We rewrite the first expectation value as

$$E_0 \left\{ \sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} (C^i(\xi^i_t) - S^*_j(t) \xi^i_t) dt \right\}$$

(46)

by subdividing the first integral. Obviously, this term is minimized by all abatement strategies $\xi^i$ fulfilling

$$\frac{\partial C^i}{\partial \xi^i}(\xi^i_t) = S^*_k(t), \quad t \in [T_{k-1}, T_k].$$

(47)

Since both an individually optimal strategy and an optimal strategy of the joint cost problem fulfill this condition (see (6) and (43)), the resulting value is the same for both strategies.

To transform the second expectation value in (45), note that for an individually optimal
strategy we have
\[ e^{-rT_k} p_k (x_{T_k}^i - Q_{i}^k) = e^{-rT_k} p_k 1_{\{x_{T_k}^i > Q_{i}^k\}} (x_{T_k}^i - Q_{i}^k) = e^{-rT_k} (S_k^* (T_k) - S_{k+1}^* (T_k)) \left( \int_0^{T_k} (y_{i}^t + n_{i}^t - \xi_{i}^t) dt - \sum_{j=1}^{k} (e_j^i + \int_0^{T_j} \theta_{j,t}^i dt) \right) \] (48)
due to (7) and we further insert the definitions of \( x_{T_k}^i \) and \( Q_{i}^k \). Using this in the second term of (45), reordering sums and integrals shows that the control variables cancel out for an individually optimal strategy and the resulting value is
\[ E_0 \left\{ \sum_{j=1}^{n} e^{-rT_j} (S_j^* (T_j) - S_{j+1}^* (T_j)) \left( \int_0^{T_j} (y_{i}^t + n_{i}^t) dt - q_{j_i}^i \right) \right\}. \] (49)
Since the optimal strategy of the global problem also fulfills (7), it results in the same value. Overall, \((\Theta^*, \Xi^*)\) solves the individual optimization problems (5), and \(S_1^*, \ldots, S_n^*\) are equilibrium permit price processes.

Finally, we show that the joint cost problem (42) corresponds to the simplified aggregate problem (18). We have seen that for a solution of (42) it holds
\[ \sum_{i \in I} (x_{T_k}^i - Q_{i}^k) = (x_{T_k} - q_k)^+. \] (50)
Further, define the aggregate abatement cost function \(C\) as
\[ C(\xi_t) = \sum_{i \in I} C_i^i(t, c^{-1}(c^i(\xi_t^i))), \] (51)
where \( j \in I \) is one arbitrarily chosen single company, \( c^i = \frac{\partial C_i^i}{\partial \xi_t^i} \) is the first derivative of \( C_i^i \), \( c^{-1} \) is its inverse function, and \( \xi_t^i \) is implicitly defined through \( \xi_t = \sum_{i \in I} c^{-1}(c^i(\xi_t^i)). \) \( C \) is well-defined because, first, \( c^i(\xi_t^i) \) is equal for all companies \( i \in I \) for a solution of (42) according to (43), and second, \( c^i \) is strictly increasing and thus invertible due to the convexity of \( C_i^i \).

Together with the differentiability of \( C_i^i \), it follows that \( C \) is convex and differentiable with
respect to $\xi_t$ and it is
\[
\sum_{i \in I} C^i(\xi_t^i) = C(\xi_t). \tag{52}
\]
Therefore, a solution of (42) corresponds to a solution of the problem (18) acting on aggregate volumes. Inverting the process, one can also recover a solution of (42) from a solution of (18), such that altogether it is equivalent to study the problem (42) or (18). The definition of $C$ and (43) also directly yields (19).

\section{Derivation of Characteristic PDEs}

We decompose the stochastic optimal control problem (18) into $n$ simpler problems as in Appendix A, that is we consider the system
\[
V_k(t, x_t, y_t) = \min_{\xi_k} \mathbb{E}_{T_{k-1}} \left\{ \int_t^{T_k} e^{-r(s-T_k-1)} C(\xi_s) \, ds \right. \\
+ e^{-r(T_k-T_{k-1})}(p_k(x_{T_k} - q_k)^+ + V_{k+1}(T_k, x_{T_k}, y_{T_k})) \left\}, \tag{53}
\]
for $t \in [T_{k-1}, T_k]$ and $k = 1, \ldots, n$, where $V_k$ is the value function of the period $k$ problem, $V_{n+1} = 0$, and $\xi_k$ is the restriction of the strategy $\xi$ to $[T_{k-1}, T_k]$.

Each of the single optimization problems in (53) can be settled by the standard dynamic programming approach along the lines of Sethi and Thompson (2006).\footnote{See also Seifert et al. (2008).} The principle of optimality yields
\[
V_k(t, x_t, y_t) = \min_{\xi_t} \mathbb{E}_{T_{k-1}} \left\{ e^{-r(t-T_{k-1})} C(\xi_t) \, dt + V_k(t + dt, x_t + dx_t, y_t + dy_t) \right\}. \tag{54}
\]

On the other hand, by applying Itô's Lemma to $V_k(t, x_t, y_t)$ with dynamics of $x_t$ and $y_t$ as given in (9) and (10) we get
\[
\mathbb{E}_{T_{k-1}} \{dV_k\} = \left( \frac{\partial V_k}{\partial t} + \frac{\partial V_k}{\partial x}(y_t - \xi_t) + \frac{\partial V_k}{\partial y} \mu_y(t) + \frac{1}{2} \frac{\partial^2 V_k}{\partial x^2} \sigma_x^2 + \frac{1}{2} \frac{\partial^2 V_k}{\partial y^2} \sigma_y^2(t) \right) dt. \tag{55}
\]
Using this in (54) leads to the Hamilton-Jacobi-Bellman (HJB) equation

\[ 0 = \min_{\xi_t} \left\{ e^{-r(t-T_{k-1})} C(\xi_t) + \frac{\partial V_k}{\partial t} + \frac{\partial V_k}{\partial x} (y_t - \xi_t) 
+ \frac{\partial V_k}{\partial y} \mu_y(t) + \frac{1}{2} \frac{\partial^2 V_k}{\partial x^2} \sigma_x^2 + \frac{1}{2} \frac{\partial^2 V_k}{\partial y^2} \sigma_y^2(t) \right\}. \]  

(56)

By differentiating the right-hand side with respect to \( \xi_t \) and setting the derivative to zero we obtain the solution

\[ \xi_t = c^{-1}(e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x}), \]  

(57)

where \( c \) stands for \( \frac{\partial C}{\partial \xi_t} \). By inserting (57) into (56), we finally arrive at the characteristic PDE

\[ \frac{\partial V_k}{\partial t} = -e^{-r(t-T_{k-1})} C(c^{-1}(e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x})) 
- \frac{\partial V_k}{\partial x} (y_t - c^{-1}(e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x})) 
- \frac{\partial V_k}{\partial y} \mu_y(t) - \frac{1}{2} \frac{\partial^2 V_k}{\partial x^2} \sigma_x^2 
- \frac{1}{2} \frac{\partial^2 V_k}{\partial y^2} \sigma_y^2(t), \]  

(58)

and the boundary condition

\[ V_k(T_k, x_{T_k}, y_{T_k}) = e^{-r(T_k-T_{k-1})}(p_k(x_{T_k} - q_k)^+ + V_{k+1}(T_k, x_{T_k}, y_{T_k})) \]  

(59)

follows from (53).
References


