Risk Factors and Their Associated Risk Premia: An Empirical Analysis of the Crude Oil Market

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April 25, 2014

Abstract

This paper investigates the role of volatility and jump risk for the pricing and hedging of derivative instruments and quantifies their associated risk premia in the crude oil futures and option markets. We propose a unified estimation approach that uses both return data and a cross section of option prices over time to consistently estimate parameters, latent variables, and to disentangle the various risk premia. Our estimation results show that jump risk is priced with a significant premium, while no evidence for a significant market price of volatility risk exists. Empirical evidence from a pricing and hedging exercise confirms these findings.
1 Introduction

In the crude oil market, physical and non-physical traders increasingly trade in linear and non-linear derivative instruments to reduce their funding costs and default risks (see Acharya, Lochstoer, and Ramadorai [2011]) or to generate excess returns in a low interest rate environment (see Falkowski [2011]). To actively manage their complex physical and non-physical portfolios market participants require an in-depth understanding of the role of different risk factors and their associated risk premia.

Previous empirical studies have analyzed risk factors and their associated risk premia in various markets. We now understand that volatility and jump risks, along with the fundamental diffusive price risk, are the most prominent risk factors under the physical measure. However, empirical results are far less conclusive with respect to the significance of their risk premia and their role for pricing and hedging performance. Major reasons are that disentangling volatility and jump risk is difficult due to their similar impact on the risk-neutral return distribution and that complex option pricing formulas make it difficult to exploit available option market data in a computationally economic manner. Moreover, empirical evidence on volatility and jump risk premia is largely based on equity markets, and it is not at all clear whether these findings are of relevance when implementing trading and hedging strategies in a different environment such as the crude oil market. Therefore, several important questions remain unanswered: Is volatility and/or jump risk priced in the crude oil option market? If so, what are the risk premia for taking over volatility and jump risk? Are these premia reflected in delta- and delta-vega hedged portfolios?

In this paper, we address these issues for the West Texas Intermediate (WTI) crude oil market. We estimate a stochastic volatility model with jumps and its nested model specifications based on a comprehensive data set of short-dated futures and option contracts from 1985 to 2010. The volatility and jump processes are disentangled by incorporating return and option market data simultaneously in the estimation approach. We overcome the above-mentioned estimation problem by including option market information through a suitably aggregated option portfolio given by synthetic variance swap rates, instead of considering each option price separately. This significantly reduces filtering errors of latent volatility states without increasing computational time considerably.

Our empirical results show that a stochastic volatility component is required to capture strongly fluctuating implied volatility levels over time. However, volatility risk alone is

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1 See e.g. Bakshi, Cao, and Chen [1997], Bates [2000], Chernov and Ghysels [2000], Pan [2002], Bakshi and Kapadia [2003a], Jones [2003], Eraker [2004], Broadie, Chernov, and Johannes [2007], Carr and Wu [2009] or Bollerslev and Todorov [2011].
not able to reflect pronounced implied volatility smiles of short-dated option contracts. This suggests that another temporary risk factor is priced in the option market. In a stochastic volatility model with jumps, the jump component is able to reproduce pronounced implied volatility smiles of short-dated option contracts, which reduces option pricing errors substantially compared to pure stochastic volatility models. This indicates that both jump and volatility risk are reflected in crude oil option prices.

Our findings on risk premia show that jump risk is priced with a significant premium, while no significant premium is paid for taking over volatility risk in the crude oil market. This is also confirmed by investigating the hedging performance of the different model specifications. For instance, we find that even after hedging delta- and vega-risks we obtain significantly upward-biased mean hedging errors for shorted out-of-the money option positions - a cross-sectional pattern best explained by the SVJ model specification combined with a jump size volatility premium. Although not priced with a premium, we find that an active management of diffusive volatility significantly reduces the standard deviation (and therefore the risks) of hedge portfolios, thereby revealing an important unspanned volatility component. Both factors are therefore important for derivative pricing and risk management, but in rather different ways.

Our study is related to a growing body of recent research looking at volatility or jump risk premia in commodity markets. Trolle and Schwartz [2009] test term structure models with different stochastic volatility specifications in the crude oil futures market between 1990 and 2006. They find that volatility risk is largely unspanned by price risk. Consequently, traders can reduce hedging errors for a single option contract if they actively hedge volatility risk by trading in other option contracts. Indeed, their results confirm that a delta-vega hedging strategy in futures and option markets significantly reduces mean absolute hedging errors compared to a delta hedging strategy in futures markets only. While Trolle and Schwartz [2009] obtain insignificant market prices of volatility risk in their model specifications Doran and Ronn [2008] find a significant negative one in their single factor volatility model based on at-the-money option contracts between 1994 and 2004. In a model-free approach, Kang and Pan [2011] estimate a negative overall variance risk premium for different maturities.

The role of jumps for crude oil futures price dynamics has only been investigated in very few studies. Dempster, Medova, and Tang [2010] show that jump events can clearly be linked to unexpected political events and find that jump diffusion models are able to capture the distributional properties of crude oil futures price returns rather well between

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2The estimated correlation parameters between futures price and volatility innovations for all model specifications are between -0.15 and 0.15.

All of the above-mentioned empirical studies on commodity markets analyze volatility and jump risk separately under the pricing measure, even though strong empirical evidence for both risk factors exists under the physical measure (see Brooks and Prokopczuk [2011]). Without testing the robustness of previous findings, it is hard to determine to what extent they are still valid in a more general framework. Our approach is distinctly different in that we consider both stochastic volatility and price jumps in one framework. This allows us to conclude that jump risk is priced with a significant premium, while no significant premium is paid for taking over volatility risk in the crude oil market. These results take effects on active risk management and efficient investment decisions in the crude oil markets.

Methodologically, our estimation approach also sets the paper apart from the earlier literature on estimating volatility and jump risk premia. We make use of a suitably aggregated option portfolio given by synthetic variance swap rates which offers several advantages. First, we are capable of incorporating a large set of option contracts without increasing computational effort considerably. For instance, Pan [2002] and Eraker [2004] restrict their analysis on a small subset of option contracts (e.g. ATM options) and a relatively short observation period which can be problematic if crisis periods or jump events are rare. Furthermore, as pointed out in Broadie, Chernov, and Johannes [2007], an estimation procedure that considers both return- and option data tests a model’s capability to simultaneously explain both data sets and thus can help to detect misspecification. In contrast to Broadie, Chernov, and Johannes [2007] however, we do not make use of a two step estimation procedure in which model calibration is undertaken separately to the time series of returns and option contracts. Most notably, such approaches recalibrate the latent volatility states in the second estimation step which can potentially cause consistency issues. We test our approach in a simulation study and show that incorporating variance swap rates in the calibration procedure significantly reduces filtering errors of latent volatility states.

The rest of the paper is organized as follows. We start by introducing the stochastic modeling approaches and then present a novel estimation method based on return and aggregated option market data. In section 3, the different model specifications are tested with regard to their distributional properties and their pricing and hedging performances using a comprehensive data set of crude oil futures and option contracts from 1985 to 2010. Section 4 concludes.
2 Stochastic Models and Pricing Formulas

In this section, we specify the stochastic volatility model with jumps (SVJ) for the futures price dynamics under the physical and the risk-neutral measure. Further, we provide pricing formulas for European options and synthetic variance swap rates. The latter are used to enhance the filtering of latent variance states as well as to obtain estimates of risk premia.

2.1 Stochastic Models

In the SVJ model, the futures price dynamics under the physical measure is given by

\[
\begin{align*}
    df_t &= (\alpha_P f_t - \lambda_z \mu_z f_t - (e^{zt} - 1) f_t - dnf_{f,t}, \\
    dv_t &= \kappa^P_v (\theta^P_v - v_t)dt + \sigma_v \sqrt{v_t} dw^P_{v,t},
\end{align*}
\]

(1)

(2)

where \( w^P_{f,t} \) and \( w^P_{v,t} \) are correlated Wiener processes with \( d[w^P_{f,t}, w^P_{v,t}] = \rho_{f,v} dt \). The superscripts \( \mathbb{P} \) and \( \mathbb{Q} \) are used to display model parameters that can differ among the physical and the risk-neutral measure, whereas model parameters without a superscript have to be the same under both measures. The two state variables \( f_t \) and \( v_t \) denote the futures price referring to a fixed maturity date and the latent variance state at time \( t \). We assume that the market price of diffusion risk is parameterized as \( \eta_f \sqrt{v_t} \) following Broadie, Chernov, and Johannes [2007]. The jump component is modeled by a Poisson process \( n_{f,t} \) with constant jump intensity \( \lambda_z \) and (percentage) jump sizes \( z_t \) that are normally distributed with mean \( \mu^P_z \) and standard deviation \( \sigma^P_z \). We allow the mean jump size and the jump size variance to differ among both measures and restrict the jump intensity to be the same under \( \mathbb{P} \) and \( \mathbb{Q} \). The drift parameter \( \alpha^P_t \) is equal to the expected excess return (futures price risk premium) of the underlying futures price dynamics. It is given by

\[
\alpha^P_t = \lambda_z \mu^P_z - \lambda_z \mu^Q_z + \eta_f \sqrt{v_t},
\]

(3)

where \(-\lambda_z \mu^P_z = -\lambda_z (e^{\mu^P_z + 0.5(\sigma^P_z)^2} - 1)\) and \(-\lambda_z \mu^Q_z = -\lambda_z (e^{\mu^Q_z + 0.5(\sigma^Q_z)^2} - 1)\) are the jump compensators under the physical and risk-neutral measure. The variance process follows a classical square-root diffusion process with a constant long-term variance level \( \theta^P_v \), mean-reversion rate \( \kappa^P_v \), and volatility of volatility parameter \( \sigma_v \). We parameterize the market

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3 In general, the measure change for a jump process is more flexible and only requires that both jump distributions are predictable and have the same support (see Cont and Tankov [2002]).

4 In the following, \( v_t \) and \( \sqrt{v_t} \) denote the variance state and the volatility state at time \( t \), respectively.
price of (diffusion) volatility risk as $\eta_v \sigma_v^{-1} \sqrt{v}$ (see, for example, Broadie, Chernov, and Johannes [2007]). It follows then that the risk-neutral futures price dynamics is given by

$$
\begin{align*}
\frac{df}{t} &= -\lambda_z \bar{f}_t dt + \sqrt{v_t} f_t dw^Q_{f,t} + (e^{z_t} - 1) f_t d\omega_{f,t}, \\
\frac{dv}{t} &= \kappa^Q_v (\theta^Q_v - v_t) dt + \sigma_v \sqrt{v_t} dw^Q_{v,t},
\end{align*}
$$

(4)

(5)

where the risk-neutral model parameters of the variance process are given by $\kappa^Q_v = \kappa^P_v + \eta_v$ and $\theta^Q_v = \frac{\kappa^P_v}{\kappa^Q_v} \theta^P_v$.

Each component of the model specification serves to describe another characteristic of observed price changes. The diffusive part in the future price process (1) captures returns under ”normal” market conditions while the stochastic volatility process allows to capture clusters in small, medium, and large price returns. In addition, the price jump component is included in order to account for rare large absolute price returns. The model specification also nests several other popular price models, namely the The Geometric Brownian Motion Model (GB), the Merton jump diffusion model (JD), as well as Heston’s stochastic volatility model (SV), each ignoring one or more components of the full model specification. For instance, the GB model is unable to capture heavy-tailed return distributions, clusters in large returns and implied volatility smiles or skews.

### 2.2 Option Pricing

Following Bakshi and Madan [2000], we obtain pricing formulas for European option contracts under all model specifications. In what follows, we drop the $t$-subscripts from both state variables $f$ and $v$, where reasonable, in order to simplify notation. In addition, although it is easy to incorporate an affine-linear stochastic interest rate process, we do not account for interest rate uncertainty due to its minor impact on short-dated option contracts (see, for example, Casassus and Collin-Dufresne [2005] or Trolle and Schwartz [2009]).

**Lemma 1** *(European Option Price Formula)*

In the GB, JD, SV, and SVJ models, the market value of a European call option with maturity date $\tau$ and strike price $k$ on a futures contract is given by

$$
c_t(k, \tau) = e^{-r(\tau-t)} \left( f \pi_t^{(1)}(\tau) - k \pi_t^{(2)}(\tau) \right),
$$

(6)

where

$$
\pi_t^{(j)}(\tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ e^{-i\phi \ln(k)} h_t^{(j)}(\tau, f, v, \phi) \right] d\phi, \quad j \in \{1, 2\}.
$$

(7)
In Appendix A, we provide the concrete functional forms of $h_t^{(1)}(\cdot)$ and $h_t^{(2)}(\cdot)$. The above pricing formula requires the numerical calculation of a one-dimensional integral. This can become an issue if (6) must be evaluated numerous times as in simulation-based estimation methods (e.g. Markov chain Monte Carlo algorithm). In this case, the resulting computational effort can become unmanageable. We solve this problem later on by using aggregated option market information instead of multiple individual option prices.

2.3 Variance Swap Contracts

In our estimation approach, we have to fit unobservable variance states, jump events, and jump sizes to market data, where their latent nature makes it difficult to obtain robust estimates based on return data only. It is generally possible to obtain more precise estimation results if option market information is incorporated in an estimation approach. However, using option market data directly is computationally intensive (see Broadie, Chernov, and Johannes [2007]). In addition, multiple option prices referring to different strikes have to be weighted “suitably” in order to filter out the single variance state at any point in time. For that reason, we propose an estimation approach that overcomes both problems by using “variance swap rates” instead of multiple option prices. The variance swap rate $v_{s,t,\tau}$ is simply defined as the “expected average annualized quadratic variation” of the underlying risk-neutral futures price process in the time period $[t, \tau]$:

$$v_{s,t,\tau} = \frac{1}{\tau - t} E_t^Q \left[ (\sigma_{t,\tau})^2 \right].$$  \hspace{1cm} (8)

It can be calculated using two approaches: (i) a model-based approach based on the underlying risk-neutral price process and (ii) a market-based approach based on a cross section of option prices. Fortunately, there is an affine-linear relation between latent variance states and variance swap rates in the SV and SVJ models. This allows us to “filter out” latent variance states by solving simple linear equations based on variance swap rates instead of using highly non-linear option price formulas directly. Based on the SVJ model, variance swap rates depend on the latent variance and jump process as follows (see Carr and Wu [2009]):

$$v_{s,t,\tau} = \frac{1}{\tau - t} E_t^Q \left[ \left( \int_t^\tau v_s ds \right) \right] + \lambda z \int_{\mathbb{R}^6} x^2 g_{n}(x) dx$$

$$= \theta^Q_v + \variancecomponent \left( v_t - \theta^Q_v \right) + \lambda z \left( (\mu^Q_z)^2 + (\sigma^Q_z)^2 \right),$$  \hspace{1cm} (9)
where \( g_{nd} \) denotes the density function of a normal distribution with mean \( \mu_z^Q \) and standard deviation \( \sigma_z^Q \). The market-based approach to calculate variance swap rates makes use of European option contracts with a continuum of strike prices. Carr and Wu [2009] show the following relation between variance swap rates and out-of-the-money European option prices:

\[
v_{st,\tau} = \frac{2}{\tau - t} \int_0^\infty \frac{o_t(k, \tau)}{e^{-r(\tau-t)k^2}} dk + \varepsilon_{v_{sr}},
\]

where \( o_t(k, \tau) \) is the market price of an European out-of-the-money option contract with strike \( k \) and maturity \( \tau \) and \( \varepsilon_{v_{sr}} \) is the approximation error in the presence of price jumps. The error term \( \varepsilon_{v_{sr}} \) is equal to:

\[
\varepsilon_{v_{sr}} = \begin{cases} 
0, & \text{GB and SV models} \\
-2\lambda_z \left( e^{\mu_z^Q + 0.5(\sigma_z^Q)^2} - 1 - \mu_z^Q - 0.5((\mu_z^Q)^2 + (\sigma_z^Q)^2) \right), & \text{JD and SVJ models}
\end{cases}
\]

This approach for calculating variance swap rates based on option prices has become a widely used market standard. If the underlying price dynamics is “correct”, the model-based and market-based expressions (9) and (10) are equal. This can then be exploited to filter out latent variance states if the remaining model parameters are known.

**Lemma 2 (Variance Swap Rates)** There exists an affine-linear relationship between the variance swap rate and the latent variance state in the SV and SVJ models. The affine-linear relation is given by:

\[
\frac{2}{\tau - t} \int_0^\infty \frac{o_t(k, \tau)}{e^{-r(\tau-t)k^2}} dk = \theta_v^Q + \frac{1 - e^{-\kappa_v^Q(\tau-t)}}{\kappa_v^Q(\tau-t)} (v_t - \theta_v^Q) + c_z^Q,
\]

where

\[
c_z^Q = \begin{cases} 
0, & \text{SV model} \\
2\lambda_z \left( e^{\mu_z^Q + 0.5(\sigma_z^Q)^2} - 1 - \mu_z^Q \right), & \text{SVJ model}
\end{cases}
\]

It is important to keep in mind that the left-hand side of (11) must be approximated, since only a finite number of option contracts are actively traded at the market. The resulting approximation error can be accounted for by assuming that variance swap rates are observed with noise.

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3 Empirical Study

3.1 Data

The data set consists of daily settlement prices of WTI crude oil futures and option contracts traded at the Chicago Mercantile Exchange (CME) Group from the Bloomberg database. We have access to front-month futures prices from January 1, 1985 to December 31, 2010 and to option market data from January 1, 2000 to December 31, 2010. The front-month futures contract is rolled over eight days before its expiry date in order to avoid maturity effects. In addition, we skip futures price returns at rolling days from our data set in order to avoid predictable price movements. In Figure 1, we plot the historical time series of the futures price process as well as absolute futures price returns during 1985-2010.

The option price data set consists of, on average, 18 option contracts with different strike prices on every business day, where option prices below 0.05 USD are eliminated as in Trolle and Schwartz [2009]. After sorting the data, we are left with a total of 80,530 option prices over our observation period. Further, we choose the three-month Treasury bill rate as the risk-free interest rate on every business day.

The variance swap rates are calculated in three steps. First, we use the approximation approach introduced by Barone-Adesi and Whaley [1987] to convert American option

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*In detail, we take settlement prices that are determined by a “Settlement Price Committee” at the end of regular trading hours (currently 2:30 p.m. EST) as in Trolle and Schwartz [2009].*
Figure 2: variance swap rates during 2000-2010

This figure shows the time series of variance swap rates based on WTI front-month crude oil futures options between 2000 and 2010.

prices to European prices.\(^7\) Then, we calculate Black-implied volatilities for each traded option contract and interpolate and extrapolate implied volatilities for missing strike prices based on cubic splines. Third, variance swap rates are calculated based on (10). The time series of variance swap rates for our data set is shown in Figure 2.

There are three conspicuous peaks and drops in the futures prices (see Figure 1). In early 1986, OPEC members failed to agree on a production limit at a meeting in Vienna. This resulted in a price drop of more than 40 percent over the following couple of months. The Gulf War II led to a strong decline in crude oil prices during 1991. After September 2008, the front-month crude oil futures price collapsed in less than one year to a third of its previously reached highest level due to the Financial Crisis. Further, variance swap rates exhibit a large peak in the Financial Crisis and two smaller peaks in 2001 (Afghanistan War) and 2003 (Iraq War) (see Figure 2).

In Table 1, we provide the summary statistics of log-return data for the complete time period and two subsamples. The first four moments are relatively stable over time and show a clear non-normal behavior: Log-returns are moderately left-skewed, where the

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\(^7\)In general, the differences between American and European option prices are rather small for short-dated option contracts. Therefore, potential option pricing errors that arise in the Barone-Adesi and Whaley approximation approach are not large.
Table 1: summary statistics


<table>
<thead>
<tr>
<th>Time Period</th>
<th>Mean</th>
<th>Volatility</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>01/1985-12/2010</td>
<td>0.0001</td>
<td>0.0233</td>
<td>-0.7930</td>
<td>18.0154</td>
<td>-0.3841</td>
<td>0.1403</td>
</tr>
<tr>
<td>01/1985-12/1999</td>
<td>-0.0001</td>
<td>0.0228</td>
<td>-1.1815</td>
<td>28.2070</td>
<td>-0.3841</td>
<td>0.1403</td>
</tr>
<tr>
<td>01/2000-12/2010</td>
<td>0.0002</td>
<td>0.0241</td>
<td>-0.3102</td>
<td>5.9084</td>
<td>-0.1654</td>
<td>0.1334</td>
</tr>
</tbody>
</table>

skewness is less pronounced in the more recent time period from 2000-2010. In addition, log-returns exhibit significant excess kurtosis in all time periods indicating a return distribution with fat-tails.

At a first glance, we find no clear evidence for a positive or negative correlation between future prices changes and volatility movements. Instead volatility peaks coincide with both strong futures price increases (e.g. 2007/2008 (Oil Price Rally)) and declines (e.g. 2008 (Financial Crisis)) in our sample. To get a rough intuition about the risk-neutral return distribution, we compute the average implied volatility smile for our data sample. We find a mostly symmetric smile form with the lowest implied volatilities for moneyness levels slightly larger than one. This suggests that a stochastic process is required that is able to capture excess kurtosis, but no positive or negative skewness compared to a simple geometric Brownian motion model.

3.2 Estimation Approach

We develop an estimation approach that uses both return data and a cross section of option prices over time. In what follows, we first devise our estimation approach and then explain why we favor the approach over obvious alternatives such as using return data only, incorporating single option prices, and performing a two-step estimation.

A Unified Estimation Framework Incorporating Option Market Information

Our methodology has two key features: First, it builds on the Markov Chain Monte Carlo (MCMC) method, a unified estimation procedure which simultaneously estimates parameters, latent variables, and risk premia. Second, option market information synthesized in form of variance swap rates facilitates the use of information in long time series of returns and the entire cross section of option prices in a consistent and computationally

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8If we compare the skewness of crude oil and equity log-returns, we find that crude oil log-returns are moderately left-skewed ($> -1.20$) compared to equities (e.g. $\approx -2.00$ for S&P 500 (see Asgharian and Bengtsson [2006])).
feasible way. In Appendix B, we detail our estimation procedure. Incorporating variance swap rates in the MCMC algorithm changes the update step of the latent variance states, while all other conditional posterior distributions are unaffected. The reason is that option market information only has an indirect impact on physical model parameters and is completely uninformative for jump times and sizes. Within this update step we basically exploit the following linear relation between latent variance states and observable out-of-the-money option prices (see (11)):

\[
\frac{2}{\tau - t} \int_0^\infty \frac{a_t(k, \tau)}{e^{-r(\tau-t)}k^2} dk = \theta_v^Q + \frac{1 - e^{-\kappa_v^Q(\tau-t)}}{\kappa_v^Q(\tau-t)} (v_t - \theta_v^Q) + 2\lambda_z (e^{\mu_v^Q + 0.5(\sigma_v^Q)^2} - \mu_v^Q) \\
= \kappa_v^P \theta_v^P + \frac{1 - e^{-(\kappa_v^P + \eta_v)(\tau-t)}}{(\kappa_v^P + \eta_v)(\tau-t)} (v_t - \kappa_v^P \theta_v^P) + c_z^Q,
\]

where \( c_z^Q = 2\lambda_z (e^{\mu_v^Q + 0.5(\sigma_v^Q)^2} - \mu_v^Q) \), and the value of the option portfolio \( \frac{2}{\tau - t} \int_0^\infty \frac{a_t(k, \tau)}{e^{-r(\tau-t)}k^2} dk \) is denoted as non-adjusted variance swap rate.

First, the above relation reveals that the risk-neutral jump size mean \( \mu_v^Q \) and jump size volatility \( \sigma_v^Q \) parameters both have merely a constant impact on the non-adjusted variance swap rate through \( c_z^Q \). Thus, based on our aggregated option market information we can estimate \( c_z^Q \) while the individual risk-neutral jump size parameters remain unknown within in our MCMC algorithm.9 Second, and more importantly, our approach is well suited to disentangle volatility and jump risk premia. In essence, our approach exploits their different impact on (non-adjusted) variance swap rates, i.e. (i) their different impact on the term structure of non-adjusted variance swap rates and (ii) their different impact on the sensitivity between the latent variance process and non-adjusted variance swap rates. In detail, a negative market price of volatility risk leads to an increasing variance swap rate in time to maturity, while the aggregated value \( c_z^Q \) has a constant impact on variance swap rates in time to maturity. Furthermore, the sensitivity of variance swap rates to changes in the latent variance process only depends on the volatility risk premium \( \eta_v \) by means of its impact on the risk-neutral mean reversion rate \( \kappa_v^P = \kappa_v^P + \eta_v \) but is independent of \( c_z^Q \). These differences allow us to separate both risk premia based on return and non-adjusted variance swap data in the MCMC estimation approach.

**Comparison to Alternative Estimations Approaches**

We have argued for the importance of incorporating a cross section of option market prices over time to enhance the filtering of latent variance states. To gauge the impact of using variance swap rates in our estimation approach, we undertake a simulation study

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9Section 3.3 will give details on how we estimate \( \mu_v^Q \) and \( \sigma_v^Q \) based on individual option prices such that they are consistent to our estimated posterior distribution of parameters and state variables.
(see Appendix C) and compare estimation runs based on return data only to estimation runs based on both return and variance swap data. The results show that standard errors of latent variance states can be reduced by about 20 percent without increasing computational effort considerably.

Compared to estimation procedures that also take option market data into account, our approach has the advantage that option prices can be efficiently incorporated without the need to numerically calculate or approximate their model values (6). In particular, if we were to use single option prices directly in an estimation approach, we could only consider a small number of option contracts over a short time period (e.g., Eraker [2004] uses a single option contract on each trading day over a time period of four years).

The alternative approach to estimate the state variables and model parameters in two steps involves the risk of inconsistent parameter estimates which is particular important for the estimation of risk premia. For instance, recalibrating volatility states to option price data could lead to an inconsistent volatility process that is on average above or below its estimated long-term level and has a deviating mean-reversion rate. To understand the problem of two-stage calibration, consider as an example the pure SV model specification. To illustrate differences between the estimation approaches, we first estimate the SV parameters under the physical measure based on the time series of futures returns only. Next we hold those parameters constant and estimate the volatility risk premium $\eta_{v}$ by using the time series of option contracts given by the variance swap rates. Notably, we recalibrate the latent volatility states in this procedure as undertaken in Broadie, Chernov, and Johannes [2007]. Indeed, we find that the average level of the latent volatility process is considerably higher and no longer consistent to the physical long-run level $\theta_{v}^{P}$ obtained under the first estimation step. Further, these higher volatility levels lead to a volatility risk premium $\eta_{v}$ that turns out to be insignificant although we show in the next section that for the SV specification this parameter estimate is significantly negative in our consistent one-stage estimation. These findings illustrate that special care has to be taken if state variables are being recalibrated. Generally, such a procedure can cause consistency issues which may conceal or artificially create risk premia.
3.3 Estimation Results

Model Parameters and State Variables

Table 2 summarizes parameter estimates for our SVJ model and its nested specifications for the entire sample period 1985 to 2010.\(^\text{10}\) The filtered volatility states in the SV and the SVJ models are illustrated in Figure 3, the filtered jump events in the JD and SVJ specification are given in Figure 4.

First, we find clear evidence for a stochastic volatility factor: For both stochastic volatility models the volatility process is highly-persistent. The filtered volatility states are very similar,\(^\text{11}\) fluctuate between 10 and 100 percent, peak in crisis periods and revert to a long term mean-level of $\sqrt{I_\theta} \approx 36\%$.

Second, the filtered jump events in the JD model indicate a misspecification as we observe very frequent jumps ($\lambda_z \approx 34$) that cluster in crisis periods such as in 1986 (OPEC Meeting in Vienna), 1991 (Gulf War II), and 2008 (Financial Crisis) (see Figure 4). This obviously violates the assumption of independent jump arrivals. In contrast, no clusters are found in the SVJ model in which we find very rare jumps ($\lambda_z \approx 1.3$) of large magnitude ($\sigma_z^P = 0.0957$) in both directions ($\mu_z^P = -0.0201$).

\(^{10}\)For robustness we also consider two sub-periods 1985-2000 as well as 2000-2010 but obtain qualitatively similar results (see Appendix D for further details).

\(^{11}\)There are two notable exceptions: (i) the Gulf War II in 1991. Here, the extreme price movement of more than 30 percent on a single day cannot be filtered out through a jump event in the SV model. (ii) the Financial Crisis in 2008. The large variance swap rates at that time increase latent variance states more strongly in the SVJ than in the SV model, since variance swap rates react less sensitively to changes in variance states in the SV than in the SVJ model (see section 3.2 for a detailed explanation)
Figure 3: filtered volatility states in the SV and SVJ models
This figure shows the estimated volatility states for the stochastic volatility model (dashed red) and the stochastic volatility model with jumps (solid blue) for the years 1985-2010.

Figure 4: filtered jump probabilities and sizes in the JD and SVJ models
This figure shows the posterior probabilities of jump events (top) and filtered jump sizes (bottom) for the jump diffusion model (left panel) and the stochastic volatility model with jumps (right panel) at each trading day during 1985-2010.
This table reports posterior means, standard deviations (in parenthesis), and 1% to 99% credibility intervals (in square brackets) for the GB, JD, SV, and SVJ (SVJ0) models. The model parameters are estimated based on the complete time period 1985 to 2010, and correspond to annual decimals. The market price of diffusion risk is set to zero in all model specifications ($\eta_f = 0$). The market price of volatility risk is estimated in the SVJ model, whereas it is set to zero in the SVJ0 model.

Table 2: model parameter estimates for the time period 1985-2010

<table>
<thead>
<tr>
<th></th>
<th>GB</th>
<th>JD</th>
<th>SV</th>
<th>SVJ</th>
<th>SVJ0</th>
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<tr>
<td>$\lambda_z$</td>
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<td>34.1738</td>
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<tr>
<td>$\mu_z$</td>
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<td>(4.4043)</td>
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<td>(0.6293)</td>
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<tr>
<td>$\mu_z^p$</td>
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<td>-</td>
<td>-0.0241</td>
<td>-0.0201</td>
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<tr>
<td>$\sigma_z$</td>
<td>-</td>
<td>(0.0019)</td>
<td>-</td>
<td>(0.0316)</td>
<td>(0.0247)</td>
</tr>
<tr>
<td>$\sigma_z^p$</td>
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<td>0.0446</td>
<td>-</td>
<td>0.0957</td>
<td>0.6920</td>
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<td>$\sigma_z^p$</td>
<td>-</td>
<td>(0.0022)</td>
<td>-</td>
<td>(0.0240)</td>
<td>(0.0204)</td>
</tr>
<tr>
<td>$\sigma_z^p$</td>
<td>-</td>
<td>(0.0022)</td>
<td>-</td>
<td>(0.0240)</td>
<td>(0.0204)</td>
</tr>
<tr>
<td>$(\sigma_z^p)^2$</td>
<td>0.1368</td>
<td>0.0712</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_f$</td>
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<td>0.0027</td>
<td>-0.0241</td>
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<tr>
<td>$\kappa_v$</td>
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<td>-</td>
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<td>-</td>
<td>(0.6233)</td>
<td>(0.6170)</td>
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<td>-</td>
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<td>[1.8174,4.6671]</td>
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<td>-</td>
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<tr>
<td>$\sigma_v$</td>
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<td>-</td>
<td>(0.0168)</td>
<td>(0.0166)</td>
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<tr>
<td>$\sigma_v$</td>
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<td>[0.1038,0.1848]</td>
<td>[0.1007,0.1807]</td>
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<td>-</td>
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<td>$\kappa_v^Q$</td>
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<td>-</td>
<td>(0.0379)</td>
<td>(0.0438)</td>
</tr>
<tr>
<td>$\kappa_v^Q$</td>
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<td>(0.0467)</td>
<td>-</td>
<td>(0.0379)</td>
<td>(0.0438)</td>
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<tr>
<td>$\theta_v^Q$</td>
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<td>[0.7317,0.9509]</td>
<td>-</td>
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<td>-</td>
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<td>-</td>
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<tr>
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<td>[0.1848,4.8563]</td>
<td>-</td>
<td>[0.0421,0.4137]</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_v$</td>
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<td>-5.3898</td>
<td>-</td>
<td>1.7716</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_v$</td>
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<td>(1.1214)</td>
<td>-</td>
<td>(1.9400)</td>
<td>-</td>
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<tr>
<td>$\eta_v$</td>
<td>-</td>
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<td>-</td>
<td>[-2.4946,4.0242]</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_v$</td>
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<td>-0.0493</td>
<td>-</td>
<td>-0.0332</td>
<td>-0.0283</td>
</tr>
<tr>
<td>$\eta_v$</td>
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<td>(0.0045)</td>
<td>-</td>
<td>(0.0091)</td>
<td>(0.0071)</td>
</tr>
<tr>
<td>$\eta_v$</td>
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<td>[0.0590,0.0384]</td>
<td>-</td>
<td>[-0.0539,-0.0117]</td>
<td>[-0.04437,-0.0109]</td>
</tr>
</tbody>
</table>
Market Price of Risk

With our estimation technology we are able to simultaneously investigate the models’ ability to fit long time series of returns and a cross section of option data over a long time period. As we have shown, this interplay is crucial for the accurate measurement of risk premia and allows us to draw meaningful conclusions regarding the market prices of diffusion, volatility, and jump risk in the crude oil futures market.

The market price of diffusion risk \( \eta_f \) is reflected in the drift component of the physical futures price process (1) via the futures price risk premium (3), but does not appear in the risk-neutral futures price process (4). The reason is that standard no-arbitrage arguments uniquely determine the risk-neutral drift component of traded futures contracts independently of the underlying data set. Thus, the market price of diffusion risk can be estimated based on historical excess return data only. In all model specifications, we find neither a significant excess return nor a significant relation between variance states and excess returns.\(^\text{12}\) This suggests that no premium is paid for taking over diffusion risk in crude oil futures markets. Henceforth, we therefore presume a market price of diffusion risk of zero in order to increase the robustness of our estimation results.

Our estimation results for the volatility and jump risk premium show that a significant aggregated market price of jump risk \( c^P_z - c^Q_z \) exists in both jump models, whereas a significant market price of volatility risk is only found in the pure stochastic volatility specification (see Table 2 and 8). While our conflicting results regarding the volatility risk premium align with the previous literature both on commodity and equity markets, straightforward computations indicate that the significant value in the pure SV specification is the effect of model misspecification rather than a reliable premium estimate. The reason is as follows: we observe a negative aggregated variance risk premium \( \frac{1}{\tau-t} (E^P_t[(\sigma_{t,\tau})^2] - E^Q_t[(\sigma_{t,\tau})^2]) \) derived from squared log-returns and short-dated variance swap rates. To capture this, the SV model requires a very large market price of volatility risk, since volatility risk diminishes when time to maturity reaches zero. While such an estimate fits, if anything, to our short-dated options it would also imply a strongly increasing (absolute) variance

\(^\text{12}\)In an unrestricted MCMC run that explicitly estimates \( \eta_f \), we obtain the following parameter estimates for the diffusion risk premium \( \eta_f \): 0.7881 (0.8586) for the GB model and 0.2816 (0.5425) for the SVJ model, where the values in the parenthesis correspond to the standard deviation of the respective posterior distribution. In both cases, the ex-post probability of a positive and a negative market price of diffusion risk is more than 5 percent so that no “significant” diffusion risk premium is found. For the SV model, the mean and standard deviation of the posterior distribution are given by 0.6211 (0.5425), where the positive and negative market prices of diffusion risk both have ex-post probabilities of more than 5 percent. In contrast to stochastic volatility models, we are not able to separate the market prices of diffusion and jump risk based on excess return data only in the JD model. The reason is that the aggregated jump risk premium \( \lambda_z P^P_z - \lambda_z P^Q_z \) and the market price of diffusion risk lead to a constant excess return (see (3)).
risk premium $\frac{1}{\tau-t} \left| \mathbb{E}_P^\tau \left[ (\sigma_{t,\tau})^2 \right] - \mathbb{E}^Q \left[ (\sigma_{t,\tau})^2 \right] \right|_{\text{in time to maturity } \tau-t}$ that is not empirically observed. So we find, for example, a model-implied variance risk premium that is about 10 times larger in absolute terms than its model-free empirical counterpart once we look at a time to maturity of six months. Further model-free analyses based on market data outside our initial data set clearly reveal that absolute variance risk premia do not increase in time to maturity (see also Kang and Pan [2011]). This inconsistency suggests that a pure stochastic volatility specification is clearly misspecified and indicates that another temporary risk factor, such as jump risk, is priced in the option market.

As a robustness test, we conduct a restricted estimation run with a market price of volatility risk of zero ($\eta_v = 0$) for the SVJ model (SVJ0) in order to test the impact of the market price of volatility risk on other model parameter estimates. It turns out that all parameter estimates remain largely unchanged, although we obtain slightly lower standard deviations for most of the model parameters in the SVJ0 specification (see Table 2).

The MCMC estimation results show that a jump risk premium is paid in the crude oil futures market, but the individual risk-neutral jump parameters $\mu^Q_z$ and $\sigma^Q_z$ are still unknown. To achieve a separation we exploit information contained in the implied volatility smile but simultaneously have to enforce consistency to our MCMC estimation results. In detail, we choose representative out-of-the-money option contracts for 10 moneyness categories ranging from 0.70 to 1.20 for every business day. To estimate $\mu^Q_z$ and $\sigma^Q_z$, we minimize squared differences of model and market-implied Black volatilities, given that the aggregated variance jump compensator is equal to the posterior mean of the MCMC estimate $\hat{c}_z^Q$:

$$\min_{(\mu^Q_z, \sigma^Q_z)} \sum_{i=1}^{n} \sum_{j=1}^{10} \left( iv_{i}^{\text{mod}}(k_j, \tau_i) - iv_{i}^{\text{mar}}(k_j, \tau_i) \right)^2$$

$$s.t. \quad 2\lambda_z(e^{\mu^Q_z+0.5(\sigma^Q_z)^2} - 1 - \mu^Q_z) = \hat{c}_z^Q,$$  

(12)

(13)

where $iv_{i}^{\text{mar}}(k_j, \tau_i)$ and $iv_{i}^{\text{mod}}(k_j, \tau_i)$ are the market-implied and model-implied volatilities for out-of-the-money option contracts with strike price $k_j$ in the $j$-th moneyness category and maturity $\tau_i$ and $n$ is the number of days in our sample.

The estimation results show that the risk-neutral mean jump size is very close to the physical counterpart ($\mu^P_z = -0.0027 (0.0019)$ and $\mu^Q_z = -0.005$ for the JD model and

---

13We do not consider all available option contracts on any trading day due to the fact that traded option contracts are unequally distributed among different moneyness categories. Furthermore, we choose the moneyness range from 0.70-1.20, because trading volumes outside this interval are very low in our data set.
\( \mu^P_z = -0.0201 \) (0.0247) and \( \mu^Q_z = -0.035 \) for the SVJ model, whereas the jump size variance is considerable larger than its statistical counterpart (\( \sigma^Q_z = 0.0586 \) compared to \( \sigma^P_z = 0.0442 \) (0.0022) for the JD model and \( \sigma^Q_z = 0.1743 \) compared to \( \sigma^P_z = 0.0921 \) (0.0204) for the SVJ model). This indicates a positive volatility of price jumps risk premium and a mean price jump risk premium close to zero.

Overall, the results of our unified estimation procedure allow us to draw conclusions regarding model specification and risk premia in the crude oil market. For both a pure jump diffusion model and a pure stochastic volatility model we find evidence for model misspecification. In the JD specification, this manifests in clustering of jump events while in the SV specification the strongly negative estimate for the volatility risk premium is in contradiction to the empirically observed term structure of aggregated variance risk premia. Risk premia are therefore most reliable in our general SVJ specification. Here, we find that jump risk is an important risk factor that is priced with a significant premium in the crude oil market, while no significant premium is found for diffusive price and volatility risk. In the next section, we elaborate on the model’s ability to capture the return and volatility dynamics as well as the cross section of option prices in more detail.

### 3.4 Empirical Tests

Distributional properties, the pricing performance, and hedging errors of the different model specifications reveal additional insights concerning the factors and risk premia that determine the time-series behavior of returns and option data in the crude oil futures market.

**Distributional Properties**

As a first assessment of the fitting quality of our estimated model specifications we simulate price paths to test qualitatively whether the respective models are able to capture the time series properties of historical log-returns. Figure 5 shows historical log-returns and simulated trajectories for all four model specifications. The estimated GB model is not able to capture clusters in large returns and price jumps, the JD model results in too many and too small price jumps, and the SV model has difficulty capturing single large absolute returns compared to the historical return data. In contrast, simulated price paths of the SVJ model exhibit similar time series properties as historical crude oil futures log-returns.
Figure 5: simulated price paths for the GB, JD, SV, and SVJ models

This figure shows the time series of real log-returns of the WTI front-month crude oil futures contract during 1985-2010 and one representative simulated price path for each model specification based on the estimated model parameters given in Table 2.
Next, we use quantile-quantile-plots (QQ-plots) to test the distributional properties of the different model specifications. The residuals are extracted by reformulating the discretized data-generating process (15) as follows:

$$
\varepsilon_{f,t,i} = \frac{y_{t,i} - (\lambda z_i^2 + \eta z v_{t,i}) \delta t - z_t \delta n_{f,t,i}}{\sqrt{v_{t,i} \delta t}}, \ i = 1, \ldots, n.
$$

If the underlying modeling approach is “correct”, the residuals are (approximately) normally distributed and we can test for normality by simple QQ-plots. It is important to keep in mind that we have applied a Bayesian estimation approach. Thus, more complex model specifications do not automatically perform better than simpler (nested) model specifications. The QQ-plots in Figure 6 reveal that a stochastic volatility component is required to capture non-normal behavior of log-returns of the crude oil futures contract during 1985-2010. For the jump diffusion model large negative returns are overestimated and large positive returns are underestimated. To formally test the different model specifications’ relative capability of capturing the distributional properties we use the Bayesian Deviance Information Criterion (DIC) proposed by Spiegelhalter, Best, Carlin, and van der Linde [2002]. This Bayesian information measure not only accounts for the “goodness of fit” to the data but also penalizes complexity. This makes it suited for model selection problems. The DIC scores are computed by using the simulated posterior distributions obtained from the MCMC algorithm. They are -29,705 for the GB model, -57,205 for the JD model, -131,753 for the SV model, and -140,653 for the SVJ model, where lower values translate into an overall superior model performance. The results confirm the importance of a stochastic volatility component, since the SV and the SVJ model scores are far smaller than the JD model. Moreover, the SVJ model specification performs best\footnote{In equity markets several empirical studies find strong evidence for a jump component in the volatility process (e.g. Eraker, Johannes, and Polson [2003], Eraker [2004] or Broadie, Chernov, and Johannes [2007]). We therefore check the distribution of model residuals of the filtered volatility process for strong deviations from normality but do not find strong evidence for jumps in volatility. The intuition for such a test is that the increments of the square-root-volatility process should be approximately normally distributed if the process dynamics are in line with the data (see Eraker [2004]).} although the difference to the simpler SV model is not very large.

While the above tests mainly focus on the time series behavior of crude oil futures returns the following comparisons of the pricing and hedging performance put more weight on the consistency between returns and options prices.

**Option Pricing Performance**

The option pricing performance is tested in two ways. First, we compare pricing errors between market-implied and model-implied variance swap rates. This answers the question...
whether the model specifications can capture the stochastic behavior of variance swap rates over time. As a second test, we compare pricing errors between market-implied and model-implied volatilities for different moneyness categories given that the model fits variance swap rates perfectly over time. Thus, here the focus is on the models’ ability to reproduce the shape of implied volatilities. To ensure that pricing errors arising from incorrectly estimated implied volatility levels (analyzed in the first test) are not mixed with pricing errors that arise when a model is not able to reproduce the smile or skew form of implied volatilities, we impose the additional restriction that the model-implied variance swap rate is equal to the market-implied variance swap rate on each business day.\footnote{This side condition is met by recalibrating the constant variance parameter $\sigma_f^2$ (GB and JD models) or latent variance states $\{v_t\}_{t=1}^n$ (SV and SVJ models) to the variance swap rate on every business day through relation (11) holding all other model parameters fixed.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{quantile-quantile-plots}
\caption{quantile-quantile-plots}
\end{figure}

This figure shows the quantile-quantile-plots for the GB, JD, SV, and SVJ models based on log-returns during the years 1985-2010.
In Figure 7, we plot the residuals between the model and market values for variance swap rates. On average, the GB model underestimates the variance swap rates, which is not surprising, given that no variance risk exists in the GB model. Thus, the variation of the futures price process has to be the same under the physical and risk-neutral measure. Therefore, differences between realized squared log-returns and variance swap rates cannot be captured in the GB model. In the JD model, the market-implied and model-implied average variance swap rates coincide through the aggregated market price of jump risk $c_P^z - c_Q^z$, but large pricing errors arise between constant model-implied and strongly fluctuating market-implied variance swap rates. The pricing errors are significantly reduced in the SV and SVJ model due to the stochastic variance process, and more so for the SVJ specification. 16 In addition to the pricing errors, we calculate the absolute pricing errors (in annualized variance) to assess the overall pricing performance. We obtain the following average absolute pricing errors for the different model specifications: 0.1590 (GB), 0.1501 (JD), 0.0601 (SV), and 0.0461 (SVJ). As expected, absolute pricing errors are at the lowest for the SV and SVJ models.

For our more detailed analysis of the shape of implied volatilities we calculate the differences between market-implied and model-implied volatilities for each option contract with strike price in one of the moneyness categories (0.70 to 1.20) on every business day. Table 3 provides the mean absolute pricing errors in each moneyness category during 2000-2010 while Table 4 contains the results for the two subsamples 2000-2008 (non-crisis period) and 2008-2009 (crisis period). We find that the GB, JD, and SV specification provide poor pricing performance for the moneyness categories 0.7-0.8 (out-of-the-money put options). The large pricing errors of the SV specification reveal that volatility risk alone is not able to generate enough excess kurtosis to capture market-implied volatility smiles. The poor option pricing performance of the JD model can be traced back to the unrealistically high jump intensity estimates. The estimated jump component implies frequent price jumps of small magnitude. This leads to an underestimation of tail risk and to an overestimation of at-the-money implied volatilities. In contrast, the SVJ model has pricing errors that are substantially smaller for out-of-the money put option contracts, since our rare and large jump estimates are able to generate enough excess kurtosis to capture pronounced market-implied volatility smiles. In addition, we compare the pricing errors during 01/2000-09/2008 (non-crisis period) and 09/2008-09/2009 (crisis period).

16 We observe rare spikes in the variance swap pricing errors under all model specifications which might be evidence for another risk factor such as jumps in the volatility process. We also calibrate an SVJ model specification with a volatility jump component but find that robust parameter estimates for the volatility jump component are difficult to obtain. Additionally, our filtered volatility increments of the MCMC algorithm do not show large deviations from normality (see last section). Consequently, we leave the incorporation of such a jump component for further research and restrict our analysis to the more parsimonious SVJ model specification.
We find that pricing errors are slightly larger during the Financial Crisis, but the results remain qualitatively identical in both subsamples.

Hedging Performance

Lastly, we analyze hedging errors of option contracts for our model specifications which is particularly important for physical traders who manage large hedge portfolios as well as intermediaries to quantify the risk inherent in their portfolios. Most importantly, the hedging performance provides valuable information whether the underlying stochastic process can capture the co-movement of futures and option prices.

In our empirical test, we compare the in-sample hedging performance of the different model specifications for representative options contracts in each of the moneyness categories introduced earlier. We compute hedge ratios for our models and set up two hedges for each option contract on each day. The first hedge uses a futures contract as hedge
Table 3: option pricing errors during 2000-2010
This table shows aggregated absolute pricing errors of different moneyness categories for the complete time period. The values correspond to annual decimals (for example, a pricing error of 0.0530 indicates a difference of 5.3% between market- and model-implied volatilities).

<table>
<thead>
<tr>
<th>moneyness</th>
<th>GB</th>
<th>JD</th>
<th>SV</th>
<th>SVJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70 - 0.75</td>
<td>0.1044</td>
<td>0.0818</td>
<td>0.0819</td>
<td>0.0559</td>
</tr>
<tr>
<td>0.75 - 0.80</td>
<td>0.0759</td>
<td>0.0523</td>
<td>0.0541</td>
<td>0.0384</td>
</tr>
<tr>
<td>0.80 - 0.85</td>
<td>0.0529</td>
<td>0.0370</td>
<td>0.0372</td>
<td>0.0321</td>
</tr>
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<td>0.85 - 0.90</td>
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<td>0.0263</td>
<td>0.0255</td>
</tr>
<tr>
<td>0.90 - 0.95</td>
<td>0.0216</td>
<td>0.0303</td>
<td>0.0203</td>
<td>0.0212</td>
</tr>
<tr>
<td>0.95 - 1.00</td>
<td>0.0206</td>
<td>0.0282</td>
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<td>0.0224</td>
</tr>
<tr>
<td>1.00 - 1.05</td>
<td>0.0246</td>
<td>0.0295</td>
<td>0.0213</td>
<td>0.0247</td>
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<tr>
<td>1.05 - 1.10</td>
<td>0.0281</td>
<td>0.0358</td>
<td>0.0249</td>
<td>0.0277</td>
</tr>
<tr>
<td>1.10 - 1.15</td>
<td>0.0399</td>
<td>0.0408</td>
<td>0.0361</td>
<td>0.0360</td>
</tr>
<tr>
<td>1.15 – 1.20</td>
<td>0.0520</td>
<td>0.0510</td>
<td>0.0480</td>
<td>0.0479</td>
</tr>
</tbody>
</table>

average pricing error 0.0454 0.0419 0.0370 0.0332

Table 4: option pricing errors during 01/2000-09/2008 and 09/2008-09/2009
The left table shows aggregated pricing errors for different moneyness categories before the Financial Crisis. The right figure shows the result during the Financial Crisis. The values correspond to annual decimals.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>GB</td>
<td>JD</td>
<td>SV</td>
</tr>
<tr>
<td>0.70 - 0.75</td>
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<td>0.0601</td>
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<td>0.75 - 0.80</td>
<td>0.0626</td>
<td>0.0409</td>
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<tr>
<td>0.80 - 0.85</td>
<td>0.0450</td>
<td>0.0322</td>
</tr>
<tr>
<td>0.85 - 0.90</td>
<td>0.0310</td>
<td>0.0309</td>
</tr>
<tr>
<td>0.90 - 0.95</td>
<td>0.0204</td>
<td>0.0310</td>
</tr>
<tr>
<td>0.95 - 1.00</td>
<td>0.0195</td>
<td>0.0228</td>
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<tr>
<td>1.00 - 1.05</td>
<td>0.0218</td>
<td>0.0294</td>
</tr>
<tr>
<td>1.05 - 1.10</td>
<td>0.0233</td>
<td>0.0332</td>
</tr>
<tr>
<td>1.10 - 1.15</td>
<td>0.0291</td>
<td>0.0339</td>
</tr>
<tr>
<td>1.15 - 1.20</td>
<td>0.0496</td>
<td>0.0504</td>
</tr>
</tbody>
</table>

average pricing error 0.0392 0.0370 0.0323 0.0311 0.0563 0.0518 0.0491 0.0452

At each day the hedge is constructed so that every option matches its actual price. We then calculate the hedge positions in the futures and option contracts according to the model-implied delta and delta-vega hedging strategy and analyze hedging errors given by the daily returns of the hedge portfolio minus the interest rate effect for each business day. Table 5 shows the dollar hedging errors for the different hedging strategies between 01/2000-12/2010 and 09/2008-09/2009. For the delta hedging strategy the hedging errors have similar means and standard deviations under all model specifications, and also the quantiles are very close. Thus, if the objective is simply to eliminate the position’s ex-
posure to price risk, model specification seems not to play a major role. In contrast, the hedging errors of the delta-vega hedging strategy, which actively manages volatility risk, have significantly lower standard deviations for the SV and SVJ models. This confirms our estimation result of only weakly correlated futures price and volatility innovations indicating a predominantly unspanned volatility factor.\textsuperscript{17} Regardless of whether stochastic volatility is a priced factor, the more narrow distribution of hedging errors clearly emphasizes the importance of including a stochastic volatility component for hedging practices. Not surprisingly, hedging volatility risk is even more important during turbulent market periods as the reduction in the standard deviation relative to delta-hedging rises from about 70\% for the whole sample period to about 80\% during the Financial Crisis.

The profit and loss of hedged option positions has been used to make inference on the sign of the diffusive volatility risk premium in studies on the equity markets (see for example Bakshi and Kapadia [2003a] or Bakshi and Kapadia [2003b]). The idea is that if the only risk factor that is priced with a significant premium is hedged (remains unhedged) the resulting mean hedging error should be zero (biased). We follow this idea and analyze mean hedging errors of delta- as well as delta-vega hedging portfolios for different moneyness categories and restrict our analysis on model specifications incorporating a stochastic volatility component. If diffusive volatility risk is the only priced risk factor and both delta- as well as vega-risk is hedged, we expect a mean hedging error indistinguishable from zero.\textsuperscript{18} In contrast, if jump risk is priced (in our case jump size volatility risk) we expect the mean error of delta-vega hedged out-of-the money options to be significantly larger than zero. Further, if jump risk is the dominant priced risk factor we also expect no statistically significant difference among the mean of delta- and delta-vega hedging errors.

Table 6 reports the mean dollar hedging errors for all moneyness categories over the period 01/2000-12/2010.\textsuperscript{19} In line with our results in section 3.3, there is evidence for a

\textsuperscript{17}Trolle and Schwartz [2009] come to a similar conclusion in a hedging study over the years 1996-2006 by using stochastic volatility models.

\textsuperscript{18}As there are no closed-form solutions for the expected hedging error under our considered model specifications we undertake a simulation study in order to formalize hypothesis for the expected hedging error. We analyze simulated hedging errors for the relevant moneyness range of 0.7-1.2. For the SV model we find that the delta-vega hedge portfolios produce mean hedging errors that are indistinguishable from zero for all moneyness categories whereas in the case of the SVJ model they are significantly larger than zero for out-of-the money option contracts. This can be explained by the fact that we make use of at-the-money options for vega-hedging. The larger the difference of the strike of the target- and hedging-option the more different is its exposure towards jump risk.

\textsuperscript{19}Note that we exclude the categories closest to at-the-money as they correspond to our hedging instrument.
priced jump component. We find that the mean delta-vega hedging errors are significantly positive for nearly all moneyness categories under consideration. This results holds true for both the SV and the SVJ model. The cross-section of simulated expected delta-vega hedging errors under the SVJ model (including a priced jump component) fits this pattern considerably better than what we obtain for the case of the SV model (and a non-zero volatility risk premium \( \eta \)). So, even after hedging away delta- as well as vega-risks we end up with a significant positive return for most of the moneyness categories, indicating the presence of another priced risk factor as expected. A look at the differences of the means of both hedging strategies confirms this. For both stochastic volatility models we cannot reject the null of identical means in the cross-section of moneyness categories (at the 1% significance-level). This is clear evidence against a priced diffusive volatility component and reinforces our findings of a misspecified SV model. Overall, we conclude that the results obtained from our analysis of hedging errors are consistent with our estimation results of a significant jump size volatility risk premium. Large jumps, be they positive or negative, seem to be a major concern of investors active on the short end of the crude oil futures market curve.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Delta</td>
<td>Delta-Vega</td>
</tr>
<tr>
<td>GB</td>
<td>0.0065</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>(0.1156)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>[−0.4305, 0.2699]</td>
<td>–</td>
</tr>
<tr>
<td>JD</td>
<td>0.0069</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>(0.1211)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>[−0.4567, 0.2884]</td>
<td>–</td>
</tr>
<tr>
<td>SV</td>
<td>0.0066</td>
<td>0.0034</td>
</tr>
<tr>
<td></td>
<td>(0.1177)</td>
<td>(0.0316)</td>
</tr>
<tr>
<td></td>
<td>[−0.4301, 0.2733]</td>
<td>[−0.0813, 0.0984]</td>
</tr>
<tr>
<td>SVJ</td>
<td>0.0062</td>
<td>0.0029</td>
</tr>
<tr>
<td></td>
<td>(0.1197)</td>
<td>(0.0337)</td>
</tr>
<tr>
<td></td>
<td>[−0.4510, 0.2722]</td>
<td>[−0.1023, 0.0842]</td>
</tr>
</tbody>
</table>

Table 5: dollar hedging errors during 09/2008-09/2009 and 2000-2010
This table reports dollar hedging errors under each model specification for the delta- and delta-vega-hedging strategies for the years 2000-2010 (complete time period) and 09/2008-09/2009 (crisis period). It shows means, standard deviations (in parenthesis), and 1% as well as 99% quantile (in square brackets) for the GB, JD, SV, and SVJ models.
Table 6: dollar hedging errors for different moneyness categories
This table shows the dollar hedging errors of the delta- and delta-vega hedging strategy for the period 2000-2010. The columns correspond to the mean of the delta hedging error, the delta-vega hedging error as well as the difference (Delta minus Delta-vega) of both means. "*" ("**") corresponds to significance at the 5% (1% level) for which t-statistics are computed according to Newey and West [1987].

<table>
<thead>
<tr>
<th>moneyness</th>
<th>SV Delta</th>
<th>Delta-Vega</th>
<th>Difference</th>
<th>SVJ Delta</th>
<th>Delta-Vega</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70 – 0.75</td>
<td>0.0074</td>
<td>0.0037</td>
<td>0.0037</td>
<td>0.0070</td>
<td>0.0031</td>
<td>0.0039</td>
</tr>
<tr>
<td>0.75 – 0.80</td>
<td>0.0057</td>
<td>0.0032**</td>
<td>0.0021</td>
<td>0.0044*</td>
<td>0.0026**</td>
<td>0.0021</td>
</tr>
<tr>
<td>0.85 – 0.90</td>
<td>0.0087**</td>
<td>0.0050**</td>
<td>0.0036*</td>
<td>0.0084**</td>
<td>0.0048**</td>
<td>0.0037</td>
</tr>
<tr>
<td>0.90 – 0.95</td>
<td>0.0041</td>
<td>0.0001</td>
<td>0.0040</td>
<td>0.0044</td>
<td>0.0004</td>
<td>0.0040</td>
</tr>
<tr>
<td>1.05 – 1.10</td>
<td>0.0077**</td>
<td>0.0034**</td>
<td>0.0045*</td>
<td>0.0072**</td>
<td>0.0026**</td>
<td>0.0046*</td>
</tr>
<tr>
<td>1.10 – 1.15</td>
<td>0.0091**</td>
<td>0.0044**</td>
<td>0.0046*</td>
<td>0.0077**</td>
<td>0.0033**</td>
<td>0.0044*</td>
</tr>
<tr>
<td>1.15 – 1.20</td>
<td>0.0072**</td>
<td>0.0031**</td>
<td>0.0040</td>
<td>0.0071**</td>
<td>0.0035**</td>
<td>0.0035</td>
</tr>
</tbody>
</table>

4 Conclusion and Outlook

In this paper, we investigate the role of volatility and jump risk in the crude oil futures market and shed light on their associated risk premia and implications for risk management. Our first contribution is to provide a novel approach to incorporate option market data through a suitably aggregated option portfolio in an estimation algorithm. The option portfolio is constructed in a way that its market value linearly depends on the latent variance state. Above all, this allows one to filter out latent variance states by solving a linear equation instead of using non-linear option price formulas. As a result we obtain consistent and more robust estimation results for model parameters, latent state variables, and risk premia without increasing computational time considerably compared to standard methods. The estimation method does not depend on any specific characteristic of energy markets, and thus can be applied to various other financial markets.

Our second contribution is a comprehensive empirical analysis of the crude oil futures price dynamics. The empirical results provide a refined view of the role of different risk factors for the pricing and hedging of derivative instruments. We show that a stochastic volatility component is required to capture the distributional properties of historical return data and that a jump component only leads to a slight further improvement. However, the picture changes when we compare the option pricing performance of the different model specifications. Here, we find that volatility risk remains an important factor in explaining strongly fluctuating variance swap rates over time. However, the shape of observed market-implied volatility smiles can only be reproduced by an additional jump component, while pure stochastic volatility models are not able to capture the tails of implied risk-neutral return distributions. In addition, we shed light on the compensation for taking over different risk factors. We find an insignificant market price of volatility risk
and a significant aggregated market price of jump risk in the stochastic volatility model with jumps. In pure stochastic volatility models, we obtain unreliable market prices of volatility risk that emphasize the importance of an appropriate model specification for estimating risk premia. This is confirmed in an analysis of delta- and delta-vega hedging errors. While managing diffusive volatility risk indeed allows one to significantly reduce the riskiness of hedging portfolios the risk factor does not seem to be priced with a premium. As a result, we conclude that the aggregated variance risk premia found in Trolle and Schwartz [2010] should be traced back to a non-zero market price of jump risk and not to a non-zero market price of volatility risk as suggested by Doran and Ronn [2008]. It thus seems that investors in the crude oil market are especially averse towards large price jumps.

A Option Pricing Functions

The functions \( h_t^{(1)}(.) \) and \( h_t^{(2)}(.) \) for the SVJ model at \( t = 0 \) are given by

\[
h_0^{(1)}(\tau, f, v, \phi) = \exp\left\{ -2\frac{\kappa_v^Q \theta_v^Q}{\sigma_v^2} \left[ \ln \left( 1 - \frac{(\xi_v - \kappa_v + (1 + i\phi)\rho_{f,v}\sigma_v)(1 - e^{-\xi_v \tau})}{2\xi_v} \right) \right] \right. \\
\left. - \frac{\kappa_v^Q \theta_v^Q}{\sigma_v^2} (\xi_v - \kappa_v + (1 + i\phi)\rho_{f,v}\sigma_v) \tau \\
+ \lambda_\xi (1 + \mu^Q_v)[(1 + \mu^Q_v)^{0.5i\phi(1+i\phi)(\sigma_v^2)^2 - 1}] - \lambda_\xi i\phi \mu^Q_v \tau \\
+ \frac{i\phi(1 + \rho_{f,v}\sigma_v)(1 - e^{-\xi_v \tau})}{2\xi_v^2 - (\xi_v - \kappa_v^Q + (1 + i\phi)\rho_{f,v}\sigma_v)(1 - e^{-\xi_v \tau})} v + i\phi \ln [f] \right\},
\]

\[
h_0^{(2)}(\tau, f, v, \phi) = \exp\left\{ \lambda_\xi [(1 + \mu^Q_v)^{0.5i\phi(1+i\phi)(\sigma_v^2)^2 - 1}] - \lambda_\xi i\phi \mu^Q_v \tau \\
- 2\frac{\kappa_v^Q \theta_v^Q}{\sigma_v^2} \left[ \ln \left( 1 - \frac{(\xi_v^* - \kappa_v + i\phi \rho_{f,v}\sigma_v)(1 - e^{-\xi_v^* \tau})}{2\xi_v^*} \right) \right] \right. \\
- \frac{\kappa_v^Q \theta_v^Q}{\sigma_v^2} [\xi_v^* - \kappa_v^Q + i\phi \rho_{f,v}\sigma_v] \tau \\
+ \frac{i\phi(1 + \rho_{f,v}\sigma_v)(1 - e^{-\xi_v^* \tau})}{2\xi_v^* - (\xi_v^* - \kappa_v^Q + i\phi \rho_{f,v}\sigma_v)(1 - e^{-\xi_v^* \tau})} v + i\phi \ln [f] \right\},
\]

where

\[
\xi_v = \sqrt{(\kappa_v^Q - (1 + i\phi)\rho_{f,v}\sigma_v)^2 - i\phi(1 + i\phi)\sigma_v^2} \\
\xi_v^* = \sqrt{(\kappa_v^Q - i\phi \rho_{f,v}\sigma_v)^2 - i\phi(1 + i\phi)\sigma_v^2}.
\]
If we insert $\lambda_z = 0$ in the above formulas, we obtain $h_t^{(1)}(.)$ and $h_t^{(2)}(.)$ for the pure stochastic volatility model. In the GB and JD model, $h_t^{(1)}(.)$ and $h_t^{(2)}(.)$ are given by the limes of $\sigma_v \to 0$.

B Estimation Method

In this section we provide details for our unified estimation framework. We use a Markov chain Monte Carlo (MCMC) algorithm which is a Bayesian and simulation-based estimation methodology. In a first step, we have to choose the market data that is to be considered in the estimation approach, prior distributions, an appropriate partition of the vector of interest (model parameters and state variables), and each sampling approach. Following standard literature, we use a log-transformation on the underlying stochastic process and then discretize the log futures price process through the quasi Monte Carlo method (see Eraker [2004] or Broadie, Chernov, and Johannes [2007]).

It follows then that

$$y_t = \ln f_{t+1} - \ln f_t = (-\lambda_z \mu + \eta_f v_t) \delta t + \sqrt{v_t \delta t} \epsilon_{f,t} + z_t \delta n_{f,t},$$

(15)

$$v_{t+1} - v_t = \kappa \theta_v (\theta_v - v_t) \delta t + \sigma_v \sqrt{v_t} \delta t \epsilon_{v,t},$$

(16)

where $\eta_f = \eta_f - 0.5$ and the time distance between two observations of the futures price process $\delta t$ is set equal to one. In (15) and (16), $\epsilon_{f,t}$ and $\epsilon_{v,t}$ are normally distributed random variables with zero means, standard deviations of one, and correlation parameter $\rho_{f,v}$. Further, $\delta n_{f,t}$ is a Bernoulli distributed random variable with jump probability $\lambda_z$ and $z_t$ is normally distributed with mean $\mu_z$ and standard deviation $\sigma_z$.

We generally choose uninformative prior distributions for all model parameters and state variables. The only exceptions are the jump intensity and the jump size variance parameter, where prior distributions capture our intuition that jumps are rare events that induce large returns. The concrete prior distributions are given in Table 7.

In the MCMC algorithm, we also must decide whether to sample each single model parameter and state variable sequentially or to group several ones and update them simultaneously. Liu, Wong, and Kong [1994] point out that sampling multiple highly correlated

20In the special case of the Black model, the log-transformation eliminates all discretization errors. However, if we consider more complex return distributions (e.g. JD, SV, and SVJ model), it is generally not possible to find a suitable transformation such that the modified process has normally distributed returns. Then, a discretization error arises, since, for instance, the time-continuous Poisson process is approximated with a Bernoulli random variable and/or variance returns are assumed to be normally distributed instead of chi-squared distributed.

21This means that the underlying return distributions depend on business days instead of calendar days.
model parameter | mean ($\mu$) | variance ($\sigma^2$) | shape ($\alpha$) | scale ($\beta$) | p | q | distribution type
--- | --- | --- | --- | --- | --- | --- | ---
$p_0^f$ | 0 | 1 | - | - | - | - | $\mathcal{N}(\mu, \sigma^2)$
$q_1^f$ | -0.5 | 1 | - | - | - | - | $\mathcal{N}(\mu, \sigma^2)$
$\lambda_z$ | - | - | - | - | 2 | 40 | $\mathcal{B}(p, q)$
$\mu^2_p$ | 0 | 1 | - | - | - | - | $\mathcal{N}(\mu, \sigma^2)$
$(\sigma^2_z)^p$ | - | - | 4 | 0.03 | - | - | $\mathcal{IG}(\alpha, \beta)$
$(\alpha, \beta)$ | 0 | 1 | - | - | - | - | $\mathcal{N}(\mu, \sigma^2)$
$\sigma_z^p$ | - | - | 4 | 0.0001 | - | - | $\mathcal{IG}(\alpha, \beta)$
$\sigma_z^v$ | - | - | 4 | 0.001 | - | - | $\mathcal{IG}(\alpha, \beta)$
$\rho_{f,v}$ | - | - | - | - | - | - | $\mathcal{U}([-1, 1])$

Table 7: prior distributions for the GB, JD, SV, and SVJ models

The table gives the concrete prior distributions (daily decimals) for all model parameters in the GB, JD, SV, and SVJ model. In the above table, $\mathcal{N}$ refers a normal distribution, $\mathcal{B}$ refers to a beta distribution, $\mathcal{IG}$ refers to an inverse gamma distribution, and $\mathcal{U}[-1, 1]$ refers to a uniform distribution on the interval $[-1, 1]$. Further, 0 corresponds to a vector of zeros and I to an identity matrix with dimensions $(2 \times 1)$ and $(2 \times 2)$, respectively.

model parameters or state variables at once can potentially increase convergence rates. However, posterior distributions of multiple parameters are often highly complex and of unknown form, which means that such blocks have to be updated through the Metropolis-Hastings algorithm. Unfortunately, it is difficult to find adequate proposal densities for such high-dimensional conditional posterior distributions. For that reason, we prefer a sequential sampling approach, which is the favored method in most empirical studies using comparable price dynamics (see Eraker, Johannes, and Polson [2003], Asgharian and Bengtsson [2006], Brooks and Prokopczuk [2011], and Larsson and Nossman [2011]).

Given that only return data $d = \{y_t\}_{i=1}^n$ is considered in the estimation approach, we obtain tractable conditional posterior distributions for the drift parameters of the futures price and variance processes, the jump intensity, the mean jump size, the jump size variance, as well as for jump times and jump sizes (see Asgharian and Bengtsson [2006]). In addition, the volatility of volatility parameter $\sigma_v$ is updated through an inverse gamma distribution, even though the conditional posterior distribution is only inverse gamma distributed for $\rho_{f,v} = 0$ (see Eraker, Johannes, and Polson [2003]).$^{22}$ The Metropolis-Hastings algorithm is only used to update latent variance states and the correlation parameter.$^{23}$

Now, we turn to the update steps given that log-returns and variance swap rates $d = (\{y_t\}_{i=1}^n, \{v_{s,t}, \tau\}_{i=1}^n)$ are incorporated in the estimation approach. The additional market information changes the conditional posterior distribution of the latent variance states, while all other conditional posterior distributions are unaffected. The reason is that option market information only has an indirect impact on physical model parameters and is

$^{22}$In a simulation study, we tested approximation errors under various parameter constellations. We found that approximation errors are negligible, even if $\rho_{f,v}$ is not close to zero.

$^{23}$In several other empirical studies, $\sigma_v$ and $\rho_{f,v}$ are updated simultaneously by a suitable reparameterization (see Jacquier, Polson, and Rossi [2004] or Brooks and Prokopczuk [2011]). However, we found that such an updating step is numerically unstable in our case.
completely uninformative for jump times and sizes. Thus, the decisive new aspect is how variance swap rate data impacts the update step of the latent variance state.

In short, the conditional posterior distribution for each variance state \( v_t \) can be expressed by using the Bayes theorem as follows:

\[
p(v_t | u_{-v_t}, vs, y) = \frac{p(v_t, vs, y | u_{-v_t})}{p(vs, y | u_{-v_t})} \propto p(v_t, vs, y | u_{-v_t}),
\]

where \( u_{-v_t} \) corresponds to the vector of model parameters and state variables excluding \( v_t \), \( vs \) denotes the vector of variance swap rates \( vs = \{vs_{t_i, \tau_i}\}_{i=1}^n \), and \( y \) is equal to the vector of log-return data \( y = \{y_{t_i}\}_{i=1}^n \). In order to more easily grasp the impact of the different model components on the filtering approach, we split the density function of the conditional posterior distribution \( p(v_t | u_{-v_t}, vs, y) \) into two analytically tractable components

\[
p(v_t | u_{-v_t}, vs, y) \propto p(v_t, vs, y | u_{-v_t}) \propto p(v_t, y | u_{-v_t}) p(vs | u_{-v_t}, v_t, y).
\]

In (17), \( p(v_t, y | u_{-v_t}) \) corresponds to the joint density function of the current variance state and log-returns of the futures price process, and \( p(vs | u_{-v_t}, v_t, y) \) is the likelihood function of variance swap rates conditional on all model parameters, state variables, and return data. These functions can be further simplified by integrating out all terms that do not depend on \( v_t \). It follows then that

\[
p(v_t | u_{-v_t}, vs, y) \propto p(v_t, y | u_{-v_t}) p(vs_{t_i, \tau_i} | u_{-v_t}, v_t),
\]

where \( p(vs_{t_i, \tau_i} | u_{-v_t}, v_t) \) is equal to one if no variance swap rate is available at \( t_i \) for \( i = 1, \ldots, n \). The function \( p(v_t, y | u_{-v_t}) \) provides the link between the current latent variance state to preceding and succeeding variance states and the preceding and current futures price log-returns. The functional form of \( p(v_t, y | u_{-v_t}) \) is given by (see Brooks and Prokopczuk [2011]):

\[
p(v_t, y | u_{-v_t}) \propto v_t^{-1} \exp(-\omega_1) \exp(-\omega_2 + \omega_3))
\]

32
with
\[
\omega_1 = \frac{(y_t - (-\lambda_z \mu^Q + \eta_f v_t) - z_t \delta n_{f,t})^2}{2v_t},
\]
\[
\omega_2 = \frac{(v_t - (v_{t-1} + \kappa^Q_v (\theta^Q_v - v_{t-1})) - \rho_{f,v} \sigma_v (y_{t-1} - (-\lambda_z \mu^Q + \eta_f v_{t-1}) - z_{t-1} \delta n_{f,t-1}))^2}{2(1 - \rho^2_{f,v}) \sigma^2_{v_t}},
\]
\[
\omega_3 = \frac{(v_{t+1} - (v_t + \kappa^Q_v (\theta^Q_v - v_t)) - \rho_{f,v} \sigma_v (y_t - (-\lambda_z \mu^Q + \eta_f v_t) - z_t \delta n_{f,t}))^2}{2(1 - \rho^2_{f,v}) \sigma^2_{v_t}}.
\]

The first component \(\exp(-\omega_1)\) puts more mass on large variance states \(v_t\) of the posterior distribution when large positive or negative diffusive returns \(\varepsilon_{f,t} = (y_t - (-\lambda_z \mu^Q + \eta_f v_t) - z_t \delta n_{f,t})\) are extracted from the log-return data. The second component \(\exp(-(\omega_2 + \omega_3))\) captures the time series properties of the variance process and the dependency structure between future price and variance innovations. The relative impact of both components on the overall conditional posterior distribution mainly depends on the volatility of volatility parameter \(\sigma_v\), where the preceding and succeeding variance states become more important for smaller volatility of volatility parameters.

The novel part of our estimation approach is the additional component \(p(v_{s_t,\tau_i}|u_{-v_{t_i}}, v_{t_i})\) that incorporates “forward-looking” market expectations about average variance levels into the filtering method. It is given by

\[
p(v_{s_t,\tau_i}|u_{-v_{t_i}}, v_{t_i}) = \frac{1}{\sqrt{2\pi} \sigma_e} \exp\left(-\frac{(v_{s_t,\tau_i} - v_{s_t,\tau_i}^{mod})^2}{2\sigma^2_e}\right), \tag{20}
\]

where

\[
v_{s_t,\tau_i}^{mod} = \theta^Q_v + \frac{1 - e^{-\kappa^Q_v (\tau_i - t_i)}}{\kappa^Q_v (\tau_i - t_i)} (v_{t_i} - \theta^Q_v) + \lambda_z \left((\mu^Q_z)^2 + (\sigma^Q_z)^2\right). \tag{21}
\]

It is derived from the affine-linear relation between the current variance state and the variance swap rate (see (9) and (11)) under the assumption that variance swap rates are observed with independent normally distributed error terms having zero means and standard deviations of \(\sigma_e\). Otherwise, if we assume that variance swap rates are observed without any noise, we obtain a singular (maximal informative) conditional posterior distribution.

In summary, our estimation approach allows us to link unobservable latent variance states to observable market data. This should improve the robustness of the estimation results and makes it possible to bring different sources of market information together. This is particularly important for obtaining good hedging results in a real market environment, since the hedging performance is highly dependent on the ability to capture the common
stochastic behavior of futures and option prices over time. Moreover, it reduces potential inconsistencies between historical and implied parameter estimates that can lead to spurious risk premia estimates and have a strong impact on the option pricing performance in two-stage estimation methods.

C Simulation Experiment

It seems rather obvious that incorporating an additional data source in an estimation approach should lower estimation errors, but the magnitude of improvement is unclear. For that reason, we conduct a simulation study for the SVJ model. The model parameters are selected close to those obtained by Larsson and Nossman [2011]. The concrete parameter values are as follows: \( \lambda_z = 6.3, \mu^p_z = -0.02, \sigma^p_z = 0.08, \mu^Q_z = -0.02, \sigma^Q_z = 0.16, \rho_{z,v} = 0, \theta^v_v = 0.126, \kappa^p_v = 3.78, \sigma_v = 0.756 \) and \( \eta_v = 0.24 \). Based on this parameter setup, we simulate 50 data sets of log-returns and variance swap rates consisting of 500 observations. Then, we perform two separate estimation runs for each simulated data set, one which makes use of return data only, and another one which uses both return and variance swap data.

The simulation results confirm the positive impact of using variance swap rates on estimating latent variance states. We find that incorporating variance swap rates reduces the root mean squared error between the filtered and the true variance process by about 20 percent. In addition, the standard deviation of the posterior distribution of variance states is reduced by about 20 percent. This can also be seen in Figure 8, which shows the standard errors of the latent variance states for one representative data set. Overall, our results confirm that using variance swap data can help us to produce more robust estimates of latent variance states without increasing computational time considerably (about 10 percent on average in our case).

D Subsample Analysis

To test the robustness of the estimation results under section 3.3 we fit the model specifications to two different subsamples (1985-2000 as well as 2000-2010). Note that due to data restriction we only have access to option data for 2000-2010. The resulting parameter estimates can be found in table 8. For both subsamples we find a highly persistent volatility process (for the stochastic volatility models) and jump events are once again

\[24\text{In addition, we test various parameter constellations (e.g. positive/negative correlations and/or positive/negative mean jump sizes) and obtain similar results under all scenarios.}\]
clustered for the JD model. There are only some minor differences among the parameter estimates: We obtain a slightly higher long-term volatility level for the second subsample which can probably be attributed to the Financial Crisis in 2008 (34% for 1985-2000 vs. 39% for 2000-2010). Further, we find a higher jump intensity for the JD as well as for the SVJ model for the first period which is consistent with our finding of fewer jump occurrences after 2000. Additionally, the results regarding the market prices of risk also remain very close to what we obtain for the whole observation period from 1985-2010.

Figure 8: standard errors of filtered variance states

The dashed red line shows the standard errors of latent variance states given that only return data is used in the estimation approach. The solid blue line shows the standard errors of latent variance states given that return and variance swap data is used in the estimation approach.
<table>
<thead>
<tr>
<th></th>
<th>GB</th>
<th>JD</th>
<th>SV</th>
<th>SVJ0</th>
<th>GB</th>
<th>JD</th>
<th>SV</th>
<th>SVJ0</th>
</tr>
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<tbody>
<tr>
<td>$\lambda_z$</td>
<td>39.5994</td>
<td></td>
<td>2.5374</td>
<td></td>
<td>-</td>
<td>18.4516</td>
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<td></td>
<td>(4.145)</td>
<td></td>
<td>(1.0386)</td>
<td></td>
<td>(4.1527)</td>
<td>-</td>
<td>(0.5380)</td>
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<tr>
<td>$\mu^P_z$</td>
<td>[29.8179,50.6602]</td>
<td>[0.7505,5.3148]</td>
<td></td>
<td>-</td>
<td>[10.3734,29.5747]</td>
<td>-</td>
<td>[0.1191,2.6151]</td>
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</tr>
<tr>
<td></td>
<td>(0.0022)</td>
<td></td>
<td>(0.0174)</td>
<td></td>
<td>(0.0052)</td>
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<td>(0.0387)</td>
<td></td>
</tr>
<tr>
<td>$\sigma^P_z$</td>
<td>-0.0070,0.0033</td>
<td>-0.0380,0.0472</td>
<td></td>
<td>-0.0220,0.0033</td>
<td>-0.1420,0.0585</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(0.0023)</td>
<td></td>
<td>(0.0135)</td>
<td></td>
<td>(0.0046)</td>
<td>-</td>
<td>(0.0178)</td>
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<tr>
<td>$(\sigma_f^P)^2$</td>
<td>0.1307</td>
<td></td>
<td>0.0537</td>
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<td>-</td>
<td>0.1388</td>
<td></td>
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<tr>
<td></td>
<td>(0.0031)</td>
<td></td>
<td>(0.0039)</td>
<td></td>
<td>(0.0037)</td>
<td>(0.0047)</td>
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<tr>
<td>$\rho_{f,v}$</td>
<td>-</td>
<td>-0.1350</td>
<td>-0.1068</td>
<td></td>
<td>-1.980</td>
<td>-0.1811</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>(0.0709)</td>
<td>(0.0611)</td>
<td></td>
<td>(0.0839)</td>
<td>(0.0813)</td>
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<tr>
<td>$\kappa^P_v$</td>
<td>-</td>
<td>8.6227</td>
<td>4.5191</td>
<td></td>
<td>-</td>
<td>3.1077</td>
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<tr>
<td></td>
<td></td>
<td>(1.1843)</td>
<td>(1.0623)</td>
<td></td>
<td>(1.0651)</td>
<td>(1.0784)</td>
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<tr>
<td>$\theta^P_v$</td>
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<td>0.0278,11.5287</td>
<td>2.2270,7.0970</td>
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<td>-</td>
<td>1.2772,6.3062</td>
<td>1.3951,6.4947</td>
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<tr>
<td></td>
<td></td>
<td>(0.0175)</td>
<td>(0.0177)</td>
<td></td>
<td>(0.0701)</td>
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<td>$\sigma_v$</td>
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<tr>
<td></td>
<td></td>
<td>(0.0576)</td>
<td>(0.0551)</td>
<td></td>
<td>(0.0513)</td>
<td>(0.0541)</td>
<td></td>
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</tr>
<tr>
<td>$\kappa^Q_v$</td>
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<td>0.6325,0.8819</td>
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<td>-</td>
<td>0.5480,0.7822</td>
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<tr>
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<td>-</td>
<td>-</td>
<td></td>
<td>-</td>
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<td>-</td>
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<tr>
<td>$\sigma^Q_v$</td>
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<td></td>
<td>-</td>
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<tr>
<td>$\eta_v$</td>
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<tr>
<td>$z^V - c^Q_z$</td>
<td>-</td>
<td>-</td>
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<td>0.1654,1.1382</td>
<td>0.1654,1.1382</td>
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<tr>
<td></td>
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<td>0.1654,1.1382</td>
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<tr>
<td>$c^P_z - c^Q_z$</td>
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<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>(0.0071)</td>
<td>-</td>
<td>(0.0076)</td>
<td></td>
<td>-</td>
</tr>
</tbody>
</table>

Table 8: model parameter estimates for the two subsamples 1985-1999 and 2000-2010

This table reports posterior means, standard deviations (in parenthesis), and 1% to 99% credibility intervals (in square brackets) for the GB, JD, SV, and SVJ0 models. The model parameters are estimated based on two subsample periods from 1985-1999 and 2000-2010, and correspond to annual decimals. The market price of diffusion risk is set to zero in all model specifications ($\eta_f = 0$). The market price of volatility risk is estimated in the SV model only (for estimation results regarding a non-zero $\eta_v$ see table 2 under section 3.3).
References


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